In[6]:= 

```
matA := 
```

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 1 \\ 9 & 1 & 9 & 0 & 9 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 3 & 1 \\ 0 & 2 & 0 & 2 & 0 \end{pmatrix}$$

```
CharacteristicPolynomial[matA, t]
```

$10\, t^2 + 21\, t^3 + 6\, t^4 - t^5$

## Reducing this mod 10 gives us the characteristic polynomial t^4 +6t^4 - t^5

To get some Intuition about the null ideal, let’s look at some of the powers of A:

In[15]:=

```
{MatrixForm[matA], MatrixForm[matA.matA],
 MatrixForm[matA.matA.matA], MatrixForm[matA.matA.matA.matA]}
```

In[13]:=

$$\left\{ \begin{pmatrix} 1 & 1 & 1 & 2 & 1 \\ 9 & 1 & 9 & 0 & 9 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 3 & 1 \\ 0 & 2 & 0 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 13 & 8 & 13 & 11 & 13 \\ 27 & 28 & 27 & 45 & 27 \\ 3 & 5 & 3 & 8 & 3 \\ 23 & 11 & 23 & 14 & 23 \\ 20 & 6 & 20 & 6 & 20 \end{pmatrix}, \begin{pmatrix} 109 & 69 & 109 & 98 & 109 \\ 351 & 199 & 351 & 270 & 351 \\ 59 & 30 & 59 & 39 & 59 \\ 159 & 108 & 159 & 157 & 159 \\ 100 & 78 & 100 & 118 & 100 \end{pmatrix}, \begin{pmatrix} 937 & 592 & 937 & 839 & 937 \\ 2763 & 1792 & 2763 & 2565 & 2763 \\ 427 & 285 & 427 & 412 & 427 \\ 1447 & 899 & 1447 & 1266 & 1447 \\ 1020 & 614 & 1020 & 854 & 1020 \end{pmatrix} \right\}$$

Notice that A+A^3 is zero mod 10. So, if P(M) is in the null Ideal for A:

P(M) = (M+M^3)Q(M) + R(M),

so we can break the polynomial into something times (M+M^3) plus something with degree 2 or lower with R(A) = 0. Thus we just need to see which quadratics are in the null ideal:

In[16]:=

```
F[i_] := MatrixForm[i matA]
```

In[18]:=
```
{F[1], F[2], F[3], F[4], F[5], F[6], F[7], F[8], F[9]}
```

Out[18]=

$$\left\{\begin{pmatrix} 1 & 1 & 1 & 2 & 1 \\ 9 & 1 & 9 & 0 & 9 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 3 & 1 \\ 0 & 2 & 0 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 2 & 4 & 2 \\ 18 & 2 & 18 & 0 & 18 \\ 2 & 0 & 2 & 2 & 2 \\ 2 & 4 & 2 & 6 & 2 \\ 0 & 4 & 0 & 4 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 3 & 3 & 6 & 3 \\ 27 & 3 & 27 & 0 & 27 \\ 3 & 0 & 3 & 3 & 3 \\ 3 & 6 & 3 & 9 & 3 \\ 0 & 6 & 0 & 6 & 0 \end{pmatrix},\right.$$

$$\begin{pmatrix} 4 & 4 & 4 & 8 & 4 \\ 36 & 4 & 36 & 0 & 36 \\ 4 & 0 & 4 & 4 & 4 \\ 4 & 8 & 4 & 12 & 4 \\ 0 & 8 & 0 & 8 & 0 \end{pmatrix}, \begin{pmatrix} 5 & 5 & 5 & 10 & 5 \\ 45 & 5 & 45 & 0 & 45 \\ 5 & 0 & 5 & 5 & 5 \\ 5 & 10 & 5 & 15 & 5 \\ 0 & 10 & 0 & 10 & 0 \end{pmatrix}, \begin{pmatrix} 6 & 6 & 6 & 12 & 6 \\ 54 & 6 & 54 & 0 & 54 \\ 6 & 0 & 6 & 6 & 6 \\ 6 & 12 & 6 & 18 & 6 \\ 0 & 12 & 0 & 12 & 0 \end{pmatrix},$$

$$\left.\begin{pmatrix} 7 & 7 & 7 & 14 & 7 \\ 63 & 7 & 63 & 0 & 63 \\ 7 & 0 & 7 & 7 & 7 \\ 7 & 14 & 7 & 21 & 7 \\ 0 & 14 & 0 & 14 & 0 \end{pmatrix}, \begin{pmatrix} 8 & 8 & 8 & 16 & 8 \\ 72 & 8 & 72 & 0 & 72 \\ 8 & 0 & 8 & 8 & 8 \\ 8 & 16 & 8 & 24 & 8 \\ 0 & 16 & 0 & 16 & 0 \end{pmatrix}, \begin{pmatrix} 9 & 9 & 9 & 18 & 9 \\ 81 & 9 & 81 & 0 & 81 \\ 9 & 0 & 9 & 9 & 9 \\ 9 & 18 & 9 & 27 & 9 \\ 0 & 18 & 0 & 18 & 0 \end{pmatrix}\right\}$$

In[23]:=
```
G[i_] := MatrixForm[i (matA.matA)]
```

In[24]:=
```
{G[1], G[2], G[3], G[4], G[5], G[6], G[7], G[8], G[9]}
```

Out[24]=

$$\left\{\begin{pmatrix} 13 & 8 & 13 & 11 & 13 \\ 27 & 28 & 27 & 45 & 27 \\ 3 & 5 & 3 & 8 & 3 \\ 23 & 11 & 23 & 14 & 23 \\ 20 & 6 & 20 & 6 & 20 \end{pmatrix}, \begin{pmatrix} 26 & 16 & 26 & 22 & 26 \\ 54 & 56 & 54 & 90 & 54 \\ 6 & 10 & 6 & 16 & 6 \\ 46 & 22 & 46 & 28 & 46 \\ 40 & 12 & 40 & 12 & 40 \end{pmatrix}, \begin{pmatrix} 39 & 24 & 39 & 33 & 39 \\ 81 & 84 & 81 & 135 & 81 \\ 9 & 15 & 9 & 24 & 9 \\ 69 & 33 & 69 & 42 & 69 \\ 60 & 18 & 60 & 18 & 60 \end{pmatrix},\right.$$

$$\begin{pmatrix} 52 & 32 & 52 & 44 & 52 \\ 108 & 112 & 108 & 180 & 108 \\ 12 & 20 & 12 & 32 & 12 \\ 92 & 44 & 92 & 56 & 92 \\ 80 & 24 & 80 & 24 & 80 \end{pmatrix}, \begin{pmatrix} 65 & 40 & 65 & 55 & 65 \\ 135 & 140 & 135 & 225 & 135 \\ 15 & 25 & 15 & 40 & 15 \\ 115 & 55 & 115 & 70 & 115 \\ 100 & 30 & 100 & 30 & 100 \end{pmatrix}, \begin{pmatrix} 78 & 48 & 78 & 66 & 78 \\ 162 & 168 & 162 & 270 & 162 \\ 18 & 30 & 18 & 48 & 18 \\ 138 & 66 & 138 & 84 & 138 \\ 120 & 36 & 120 & 36 & 120 \end{pmatrix},$$

$$\left.\begin{pmatrix} 91 & 56 & 91 & 77 & 91 \\ 189 & 196 & 189 & 315 & 189 \\ 21 & 35 & 21 & 56 & 21 \\ 161 & 77 & 161 & 98 & 161 \\ 140 & 42 & 140 & 42 & 140 \end{pmatrix}, \begin{pmatrix} 104 & 64 & 104 & 88 & 104 \\ 216 & 224 & 216 & 360 & 216 \\ 24 & 40 & 24 & 64 & 24 \\ 184 & 88 & 184 & 112 & 184 \\ 160 & 48 & 160 & 48 & 160 \end{pmatrix}, \begin{pmatrix} 117 & 72 & 117 & 99 & 117 \\ 243 & 252 & 243 & 405 & 243 \\ 27 & 45 & 27 & 72 & 27 \\ 207 & 99 & 207 & 126 & 207 \\ 180 & 54 & 180 & 54 & 180 \end{pmatrix}\right\}$$

First see that since the last entry is zero, we won't need any "constant" term. We need bA+cA^2, and b is nonzero since none of the cA^2 are 0. Since the first three entries of the bA are equal, the first three entries of the cA^2 should be equal, so c needs to be even. So candidates are {2A + 6A^2, 4A+2A^2, 6A+8A^2, 8A+4A^2}~{3(4A+2A^2),4A+2A^2,4(4A+2A^2), 2(4A+2A^2)}.

Hence R(M) = c(4M+2M^2).

So, the null ideal is generated by M+M^3 and 4M+2M^2.