Problem 1: Given n points $\{(a_1, b_1), \ldots, (a_n, b_n)\}$ with $x_i \neq x_j$ and $y_i \neq y_j$, for $i \neq j$. Find a polynomial who goes through all the points.

Solution: We will claim there is a polynomial of degree $n-1$ that goes through all the points. Consider such polynomial $p(x)$. We then have n equations, namely $p(a_i) = b_i$ for $1 \leq i \leq n$. This system of equations can be written in matrix form:

$$\underbrace{\begin{bmatrix} 1 & a_1 & a_1^2 & a_1^3 & \ldots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & a_2^3 & \ldots & a_2^{n-1} \\ 1 & a_3 & a_3^2 & a_3^3 & \ldots & a_3^{n-1} \\ \vdots & & \vdots & & \ddots & \vdots \\ 1 & a_n & a_1^2 & a_n^3 & \ldots & a_n^{n-1} \end{bmatrix}}_{=\mathbf{A}} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

The polynomial we want is: $p(x) = \sum_{i=0}^{n-1} c_i x^k$ where the coefficients c_i's are given by:

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & a_1 & a_1^2 & a_1^3 & \ldots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & a_2^3 & \ldots & a_2^{n-1} \\ 1 & a_3 & a_3^2 & a_3^3 & \ldots & a_3^{n-1} \\ \vdots & & \vdots & & \ddots & \vdots \\ 1 & a_n & a_1^2 & a_n^3 & \ldots & a_n^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

The matrix $\mathbf{A}$, sometimes called the Vandermonde matrix, has its determinant:

$$\det(\mathbf{A}) = \prod_{1 \leq i < j \leq n} (a_j - a_i).$$

Since all the a_k's are different, the product, and thus, the determinant is non-zero. So, $\mathbf{A}$ is in fact invertible.

Problem 2: Derive the closed form formula for the fibonacci numbers f_k where $f_0 = f_1 = 1$ and $f_n = f_{n-2} + f_{n-1}$ for $n \geq 2$.

Solution (Generating Function): Consider the generating function:

$$\begin{aligned} G(x) = \sum_{n=0}^{\infty} f_n x^n = 1 + x + \sum_{n=2}^{\infty} f_n x^n = 1 + x + \sum_{n=2}^{\infty} (f_{n-1} + f_{n-2}) x^n \\ = 1 + x + \sum_{n=2}^{\infty} f_{n-1} x^n + \sum_{n=2}^{\infty} f_{n-2} x^n \end{aligned}$$

Examine the two sums:

$$\sum_{n=2}^{\infty} f_{n-1}x^n = \sum_{n=1}^{\infty} f_n x^{n+1} = x\sum_{n=1}^{\infty} f_n x^n = x\big(G(x)-1\big)$$
$$\sum_{n=2}^{\infty} f_{n-2}x^n = \sum_{n=0}^{\infty} f_n x^{n+2} = x^2\sum_{n=0}^{\infty} f_n x^n = x^2 G(x).$$

We can now rewrite $G(x)$ as:

$$G(x) = 1 + x + xG(x) - x + x^2G(x) \Longrightarrow G(x) = \frac{1}{1-x-x^2}$$

Factor the denominator: $1 - x - x^2 = (1-r_1x)(1-r_2x)$ where $r_1 = \frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{1-\sqrt{5}}{2}$. So, we can rewrite the expression and apply partial fraction:

$$G(x) = \frac{1}{(1-r_1x)(1-r_2x)} = \frac{1}{r_1-r_2}\left(\frac{r_1}{1-r_1x} - \frac{r_2}{1-r_2x}\right)$$
$$= \frac{1}{\sqrt{5}}\left(\frac{r_1}{1-r_1x} - \frac{r_2}{1-r_2x}\right).$$

Lastly, we need to convert the expression back into a power series using this fact:

$$\frac{a}{1-ax} = \sum_{n=0}^{\infty} a^{n+1}x^n.$$

We can rewrite $G(x)$ as:

$$G(x) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}}(r_1^{n+1} - r_2^{n+1})x^n$$

Therefore, $f_n = \frac{1}{\sqrt{5}}(r_1^{n+1} - r_2^{n+1})$, or,

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1+\sqrt{5}}{2}\right)^{n+1}\right]$$

Solution (Difference Equation): The fibonaaci series is a homogenous second order difference equation with the characteristic equation of:

$$r^2 - r - 1 = 0.$$

The solutions to the quadratic are given by:

$$r_1 = \frac{1+\sqrt{5}}{2} =: \phi, \quad \text{and} \quad r_2 = \frac{1-\sqrt{5}}{2} = 1-\phi.$$

This implies the general solution to be $f_n = k_1\phi^n + k_2(1-\phi)^n$. We have $f_0 = 1 = k_1 + k_2$, and $f_1 = 1 = k_1\phi + k_2(1-\phi)$. Solving for k_1, k_2 we have:

$$k_1 = \frac{\phi}{2\phi-1}; \quad \text{and} \quad k_2 = \frac{\phi-1}{2\phi-1}.$$

Therefore, we have:

$$f_n = \frac{\phi^{n+1} - (1-\phi)^{n+1}}{2\phi-1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$