PROBLEM OF THE WEEK - FALL 2014 - WEEK OF OCTOBER 27

Question 1

Let's reword a classic problem: Say there are k coins in a container. When you try to take exactly 2 coins out at a time you keep going until you are left with 1 of them and so you can't remove it. When you try this again by taking coins out 3 at a time, you can keep removing until you end up with 2 coins, i.e., k has a remainder when you try to divide it by 3. You try this process again, removing coins 4 at a time then 5 at a time then 6 at a time, but with each of these attempts you have coins left over: 3 then 4 then 5, respectively. Finally, you remove the coins 7 at a time and you can do this until none are left: 7 divides k without remainder.

Can you tell exactly what k is? If not, what's the smallest number k can be?

Question 2

Let R be a ring (with unit $1 \neq 0$ and not necessarily commutative). Suppose there are elements $a, b, c \in R$ such that:

(i) $ab = ba$ and $bc = cb$,

(ii) For all $x, y \in R$, $bx = by$ implies $x = y$,

(iii) $ca = b$ but $ac \neq b$.

Show that R must have infinitely many elements.