A differential equation is an equation that contains an unknown function together with one or more of its derivatives.

\[ x^2 - 5x + 6 = 0 \]  
(Here, the unknown is a number)
Examples:

1. \( y' = 2x + \cos x \)

2. \( \frac{dy}{dt} = ky \) (exponential growth/decay)

3. \( x^2 y'' - 2xy' + 2y = 4x^3 \)
4. \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \) (Laplace’s eqn.)

5. \( \frac{d^3 y}{d x^3} - 4 \frac{d^2 y}{d x^2} + 4 \frac{d y}{d x} = 0 \)

6. \( \frac{\partial^2 u}{\partial y^2} = k \frac{\partial^2 u}{\partial x^2} \) (wave equation)
**TYPE:**

If the unknown function depends on a single independent variable, then the equation is an **ordinary differential equation (ODE)**; if the unknown function depends on more than one independent variable, then the equation is a **partial differential equation (PDE)**.
ORDER:

The order of a differential equation is the order of the highest derivative of the unknown function appearing in the equation.
Examples:

1. \[ \frac{dy}{dt} = k y \]  
   1st order ordinary

2. \[ x^2 y'' - 2xy' + 2y = 4x^3 \]  
   2nd order Ordinary

3. \[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]  
   3rd order partial
4. \[ \frac{d^3 y}{dx^3} - 4 \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} = 0 \]

3rd ordinary

5. \[ \frac{d^2 y}{dx^2} + 2x \sin \left( \frac{dy}{dx} \right) + 3e^{xy} = \frac{d^3}{dx^3} (e^{2x}) \]

2nd order ordinary

6. \[ \frac{\partial^2 u}{\partial y^2} = k \frac{\partial^2 u}{\partial x^2} \]

2nd partial
SOLUTION:

A solution of a differential equation is a function defined on some domain $D$ such that the equation reduces to an identity when the function is substituted into the equation.
Examples:

1. \( y' = 2x + \cos x \)

\[
\begin{align*}
  y &= x^2 + \sin x \\
  y' &= 2x + \cos x \\
  2x + \cos x &= 2x + \cos x
\end{align*}
\]

\( y \leq x^2 + \sin x + C \) or \( \text{Soln for any constant } C \)
2. \( y' = ky \)\\

Exponential func\\
\( y = e^{bt} \)

Check \( y' = ke^{bt} \)

\( ke^{bt} \neq b(e^{bt}) = be^{bt} \)

\( y = Ce^{bt} \) is a soln for any constant \( C \)

Set of all solutions
3. \( x^2 y'' - 2xy' + 2y = 4x^3 \)

Is \( y = x^2 + 2x^3 \) a solution? \( \text{Yes} \)

\[
y = x^2 + 2x^3
\]
\[
y' = 2x + 6x^2
\]
\[
y'' = 2 + 12x
\]

\[
x^2 (2 + 12x) - 2x (2x + 6x^2) + 2x^3
\]

\[
2 (x^2 + 2x^3) = 4x^3
\]

\[
2x + 12x - 4x^2 - 12x + 2x^2 + 4x = 4x^3
\]

\[
4x^3 = 4x^3
\]
4. \[ x^2 y'' - 2xy' + 2y = 4x^3 \]

Is \( y = 2x + x^2 \) a solution? \( \text{No} \)

\begin{align*}
y &= 2x + x^2 \\
y' &= 2 + 2x \\
y'' &= 2
\end{align*}

\begin{align*}
x(2)^2 - 2x(2 + 2x) + 2(2x + x^2)^2 &\quad = 4x^2 \quad \checkmark \\
2x^2 - 4x - 4x^2 + x^2 + 2x &\quad = 4x^3 \\
0 &\quad \neq 4x^3
\end{align*}
4. \[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]

\[ u = \ln \sqrt{x^2 + y^2} \quad \text{Solution?} \]

\[ = \ln \left( x^2 + y^2 \right)^{1/2} = \frac{1}{2} \ln \left( x^2 + y^2 \right) \]

\[ \frac{\partial u}{\partial x} = \frac{1}{2} \cdot \frac{1}{x^2 + y^2} \cdot 2x = \frac{x}{x^2 + y^2} \]

\[ \frac{\partial^2 u}{\partial x^2} = \left( \frac{x^2 + y^2}{(x^2 + y^2)^2} \right) \frac{1 - x(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \]

\[ \frac{\partial^2 u}{\partial y^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2} \]

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]
5. \[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]

\[ u = \cos x \sinh y, \quad u = 3x - 4y \]

Solutions??

Yes - you can check this.
6. Find values of $r$ such that $y = e^{rx}$ is a solution of $e^x, e^{-2x}$ are solns

$$y'' - 3y' - 10y = 0.$$ 

We want $y = e^{nx}$ to be a soln.

\[
y = e^{nx}, \\
y' = ne^{nx}, \\
y'' = n^2 e^{nx}
\]

\[n^2 e^{nx} - 3(n e^{nx}) - 10(e^{2x}) = 0\]

\[e^{nx}((n^2 - 3n - 10) = 0)\]

\[n^2 - 3n - 10 = 0\]

\[(n - 5)(n + 2) = 0\]

\[n = 5, n = -2\]
7. Find values of $r$ such that $y = x^r$ is a solution of

$$x^2 y'' - 5x y' + 8y = 0.$$
8. Find values of $r$ such that $y = x^r$ is a solution of

$$y'' - \frac{6}{x^2} y = 0.$$ 

$y = x^r$

$y' = nx^{r-1}$

$y'' = n(n-1)x^{r-2}$

$n(n-1)x^{r-2} - \frac{6}{x^2} x^r = 0$ for all $x$.

$x^{r-2} \left[ n^2 - n - 6 \right] = 0$

$n^2 - n - 6 = 0$

$(n-3)(n+2) = 0$

$n = 3, n = -2$

$y_1 = x^3, y_2 = x^{-2}$
9. Find values of $A$ such that

$y = Ae^{2x}$ is a solution of

$$y'' - y' - 6y = 8e^{2x}.$$ 

$y = Ae^{2x}$

$y' = 2Ae^{2x}$

$y'' = 4Ae^{2x}$

$4Ae^{2x} - (2Ae^{2x}) - 6Ae^{2x} = 8e^{2x}$

$-4Ae^{2x} = 8e^{2x}$

$A = -2$

$y = -2e^{2x}$ is a sol.
From now on, all differential equations are ordinary differential equations.
n-PARAMETER FAMILY OF SOLUTIONS:

Example: Solve the differential equation:

$$y''' - 12x + 6e^{2x} = 0$$

$$y''' = 12x - 6e^{2x} \quad \frac{c_1}{x} \quad c_1$$

$$y'' = 6x^2 - 3e^{2x} + C_1$$

$$y' = 2x^3 - \frac{3}{2}e^{2x} + C_1x + C_2$$

$$y = \frac{x^4}{2} - \frac{3}{4}e^{2x} + C_1x^2 + C_2x$$

$$+ C_3$$

3 para. family
Intuitively, to find a set of solutions of an \( n \)-th order differential equation we “integrate” \( n \) times, with each integration step producing an arbitrary constant of integration. Thus, ”in theory,” an \( n \)-th order differential equation has an \( n \)-parameter family of solutions.
SOLVING A DIFFERENTIAL EQUATION:

To solve an $n$-th order differential equation means to find an $n$-parameter families of solutions. (Note: Same $n$.)
Examples: $n$-parameter family of solutions:

1. $y' = 3x^2 - 2x + 4$

   $y = x^3 - x^2 + ux + C$

   1 parameter family
2. $y'' = 2x + \sin 2x$

$y' = x^2 - \frac{1}{2} \cos 2x + C_1$ 

$y = \frac{1}{3} x^3 - \frac{1}{4} \sin 2x + C_1 x + C_2$

2 param family general soln.
3. \[ y''' - 3y'' + 3y' - y = 0 \]

Answer: \[ y = C_1 e^x + C_2 xe^x + C_3 x^2 e^x \]

4. \[ x^2 y'' - 2xy' + 2y = 4x^3 \]

Answer: \[ y = C_1 x + C_2 x^2 + 2x^3 \]
GENERAL SOLUTION/SINGULAR SOLUTIONS:

An “$n$-parameter family of solutions” is also called the general solution.

Solutions of an $n$-th order differential equation which are not included in an $n$-parameter family of solutions are called singular solutions.
Example:

\[
\frac{dy}{dx} = (4x + 2)(y - 2)^{1/3}
\]

General solution:

\[
(y - 2)^{2/3} = \frac{4}{3} x^2 + \frac{4}{3} x + C
\]

1. Parametric family (Section 2.2)

Singular solution: \( y \equiv 2 \)

\[
y = 2, \quad y' = 0, \quad 0 = (4x + 2)(2 - 2)^{1/3} = 0
\]

Set of all solutions:

\[
(y - 2)^{2/3} = \frac{4}{3} x^2 + \frac{4}{3} x + C \quad \text{plus:} \quad y \equiv 2
\]
PARTICULAR SOLUTION:

If specific values are assigned to the arbitrary constants in the general solution of a differential equation, then the resulting solution is called a particular solution of the equation.
Examples:

1. \( x^2 y'' - 2xy' + 2y = 4x^3 \)

General solution:

\[
y = C_1 x + C_2 x^2 + 2x^3
\]

Particular solutions:

Set \( C_1 = C_2 = 0; \quad y = 2x^3 \)

Set \( C_1 = 3, \quad C_2 = -2; \)

\[
y = 3x - 2x^2 + 2x^3
\]
2. \[ \frac{dy}{dx} = (4x + 2)(y - 2)^{1/3} \]

General solution:

\[ (y - 2)^{2/3} = \frac{4}{3}x^2 + \frac{4}{3}x + C \]

Particular solutions:

\( C = 0 : \quad (y - 2)^{2/3} = \frac{4}{3}x^2 + \frac{4}{3}x \)

\( C = -5 : \quad (y - 2)^{2/3} = \frac{4}{3}x^2 + \frac{4}{3}x - 5 \)

Singular solution: \( y \equiv 2 \)
THE DIFFERENTIAL EQUATION OF AN $n$-PARAMETER FAMILY:

Given an $n$-parameter family of curves. The differential equation of the family is an $n$-th order differential equation that has the given family as its general solution.
Examples: 1. \( y^2 = Cx^3 + 4 \) is the general solution of a DE.

a. What is the order of the DE? \( \text{one} \)

b. Find the DE?

\[
y^2 = Cx^3 + 4
2yy' = 3Cx^2
C = \frac{2y'y'}{3x^2}
\frac{y^2}{2} - \frac{12}{x^2} = \frac{2yy'y'}{3x^2}
\]
2. \[ y = C_1 e^{2x} + C_2 e^{3x} \] is the general solution of a DE.

a. What is the order of the DE?

\[ 2\text{nd order} \]

b. Find the DE?

\[
\begin{align*}
y &= C_1 e^{2x} + C_2 e^{3x} \\
y' &= 2C_1 e^{2x} + 3C_2 e^{3x} \\
y'' &= 4C_1 e^{2x} + 9C_2 e^{3x}
\end{align*}
\]

\[
\begin{align*}
y &= C_1 e^{2x} + C_2 e^{3x} \\
y' &= 2C_1 e^{2x} + 3C_2 e^{3x} \\
y'' &= 4C_1 e^{2x} + 9C_2 e^{3x}
\end{align*}
\]

\[ y' - 2y = C_2 e^{3x} \]

\[ C_2 = \frac{y' - 2y}{e^{3x}} \]
\[ y'' - 3y' = -C_1 e^{2x} \]

\[ C_1 = -\left( \frac{y'' - 3y'}{e^{2x}} \right) \cdot 3y - y' \]

\[ y'' = 4 \left( \frac{3y - y'}{e^{2x}} \right) e^{2x} + 9 \left( \frac{y' - 2y}{e^{3x}} \right) e^{3x} \]

\[ y''' = 4 \left( 3y - y' \right) + 9 \left( y' - 2y \right) \]

\[ y'' - 5y' + 6y = 0 \]

This is the differential equation.
Strategy for finding the differential equation

Step 1. Differentiate the family $n$ times. This produces a system of $n + 1$ equations.

Step 2. Choose any $n$ of the equations and solve for the parameters. (constants)

Step 3. Substitute the “values” for the parameters in the remaining equation.
Examples:

The given family of functions is the general solution of a differential equation.

(a) What is the order of the equation?

(b) Find the equation.
1. \[ y = C_1 x^2 + C_2 x^3 + C_3 \]

(a) 3rd order
deq

(b) \[ y = C_1 x + C_2 x^2 + C_3 \]
\[ y' = 2C_1 x + 3C_2 x^2 \]
\[ y'' = 2C_1 + 6C_2 x \]
\[ y''' = 6C_2 \quad C_2 = \frac{y''}{6} \]
\[ y'''' = 2C_1 + 6 \left( \frac{y''}{6} \right) x \]
\[ = 2C_1 + xy''' \]

\[ C_1 = y - \frac{xy'''}{2} \]
\[ y' = 2 \left( \frac{y - xy'''}{2} \right) x + 3 \left( \frac{y''}{6} \right) x^2 \]
2. \[ y = C_1 \cos 2x + C_2 \sin 2x \]

(a) \[ 2\text{nd order} \]

(b) \[ y = C_1 \cos 2x + C_2 \sin 2x \]
\[ y' = -2C_1 \sin 2x + 2C_2 \cos 2x \]
\[ y'' = -4C_1 \cos 2x - 4C_2 \sin 2x \]
\[ = -4 \left( C_1 \cos 2x + C_2 \sin 2x \right) \]
\[ = -4y \]
\[ y' = -4y \]
\[ y'' = -4y \]
\[ y'' + 4y = 0 \]
3. \( y = C_1 + C_2x + C_3x^2 \)

(a) 3rd order

(b) \( y = C_1 + C_2x + C_3x^2 \)

\[ y' = C_2 + 2C_3x \]

\[ y'' = 2C_3 \]

\[ y''' = 0 \quad \text{done} \]

\[ y = C_1 + C_2x + C_3x^2 \]

\[ y'' = 0 \]

\[ y = C_1x + C_2 \]

\[ y_1 = C_1 \]

\[ y_2 = \text{solution} \]

\[ y = C_1x^2 + C_2x + C_3 \]
1. Find a solution of

\[ y' = 3x^2 + 2x + 1 \]

which passes through the point \((-2, 4)\).
2. \( y = C_1 \cos 3x + C_2 \sin 3x \) is the general solution of

\[ y'' + 9y = 0. \]

a. Find a solution which satisfies

\[ y(0) = 3 \]

\[ y(0) = C_1 \cos 0 + C_2 \sin 0 = 3 \]

\[ C_1 = 3 \]

\[ y = 3 \cos 3x + C_2 \sin 3x \] satisfies for all \( C_2 \).
b. Find a solution which satisfies

\[ y(0) = 4, \quad y(\pi) = 4 \]

\[ y = C_1 \cos 3x + C_2 \sin 3x \]

\[ y(0) = C_1 \cos 0 + C_2 \sin 0 = C_1 = 4 \]

\[ y(\pi) = C_1 \cos \pi + C_2 \sin \pi = -C_1 = 4 \]

\[ C_1 = -4 \]

No soln. This is called a two point boundary pb.
c. Find a solution which satisfies

\[ y(\pi/4) = 1, \quad y'(\pi/4) = 2 \]

\[ y = C_1 \cos 3x + C_2 \sin 3x \]

\[ y(\pi/4) = C_1 \left(-\frac{\sqrt{2}}{2}\right) + C_2 \left(\frac{\sqrt{2}}{2}\right) = 1 \]

\[ y' = -3 C_1 \sin 3x + 3 C_2 \cos 3x \]

\[ y'(\pi/4) = C_1 \left(-3 \frac{\sqrt{2}}{2}\right) - C_2 \left(\frac{3\sqrt{2}}{2}\right) = 2 \]

\[ -\frac{\sqrt{2}}{2} C_1 + \frac{\sqrt{2}}{2} C_2 = 2 \]

\[ -3 \sqrt{2} C_1 - 3 \sqrt{2} C_2 = 4 \]

\[ -6 \sqrt{2} C_1 = 10 \]

\[ C_1 = -\frac{10}{6 \sqrt{2}} = -\frac{5}{3 \sqrt{2}} \]

\[ C_2 = \frac{\sqrt{2}}{2} \]

\[ C_1 = \frac{\sqrt{2}}{2} \]

\[ C_2 = \frac{\sqrt{2}}{2} \]

\[ C_2 = \frac{\sqrt{2}}{2} \]

\[ C_2 = \frac{\sqrt{2}}{2} \]

\[ C_2 = \frac{\sqrt{2}}{2} \]

\[ C_2 = \frac{\sqrt{2}}{2} \]

\[ C_2 = \frac{\sqrt{2}}{2} \]
\[-\sqrt{2} C_1 + \sqrt{2} C_2 = 2 \quad \text{\underline{\text{-3}}} \]

\[-3 \sqrt{2} C_1 - 3 \sqrt{2} C_2 = 4 \]

\[-6 \sqrt{2} C_2 = -2 \quad C_2 = \frac{1}{3 \sqrt{2}} \]

\[y = -\frac{5}{3 \sqrt{2}} \cos 3x + \frac{1}{3 \sqrt{2}} \sin 3x \]
An \textit{n-th order initial-value problem} consists of an \textit{n-th order differential equation}

\[ F \left[ x, y, y', y'', \ldots, y^{(n)} \right] = 0 \]

together with \textit{n} (initial) conditions of the form

\[ y(c) = k_0, \ y'(c) = k_1, \ y''(c) = k_2, \ldots, \]

\[ y^{(n-1)}(c) = k_{n-1} \]

where \( c \) and \( k_0, \ k_1, \ldots, \ k_{n-1} \) are given numbers.
NOTES:

1. An $n$-th order differential equation can always be written in the form

$$F \left[ x, y, y', y'', \ldots , y^{(n)} \right] = 0$$

by bringing all the terms to the left-hand side of the equation.

2. The initial conditions determine a particular solution of the differential equation.
Strategy for Solving an Initial-Value Problem:

Step 1. Find the general solution of the differential equation.

Step 2. Use the initial conditions to solve for the arbitrary constants in the general solution.
Examples:

\[ y' = 5x - 3x^3 \]

\[ y = C_1x + C_2x^3 \] is the general solution of

\[ y'' - \frac{3}{x} y' + \frac{3}{x^2} y = 0 \]

a. Find a solution which satisfies

\[ y(1) = 2, \quad y'(1) = -4. \]
b. Find a solution that satisfies

\[ y(0) = 0, \quad y'(0) = 2. \]

\[ y = c_1 x + c_2 x^3 \]
\[ y' = c_1 + 3c_2 x^2 \]

\[ y(0) = c_1 (0) + c_2 (0) = 0 \]

swell \( c_1 = 0 \) all \( c_1, c_2 \) satisfy this

\[ y'(0) = c_1 + 0 = 2 \]
\[ c_1 = 2 \]

The problem is: \( 0: y'' - \frac{3}{x} y' + \frac{3}{x^2} y = 0 \)

You can't let \( x = 0 \) in the d.e.
c. Find a solution that satisfies

\[ y(0) = 2, \quad y'(0) = 3. \]

\[ y = C_1 x + C_2 x^2, \]
\[ y' = C_1 + 2C_2 x. \]

\[ y(0) = C_1 (0) + C_2 (0) = 2 \]
\[ 0 = 2 \]

false
EXISTENCE AND UNIQUENESS:

The fundamental questions in a course on differential equations are:

1. Does a given initial-value problem have a solution? That is, do solutions to the problem exist?

2. If a solution does exist, is it unique? That is, is there exactly one solution to the problem or is there more than one solution?