A second order linear differential equation is an equation which can be written in the form

$$y'' + p(x)y' + q(x)y = f(x)$$

where $p$, $q$, and $f$ are continuous functions on some interval $I$.

The functions $p$ and $q$ are called the coefficients of the equation.
The function $f$ is called the \textbf{forcing function} or the \textbf{nonhomogeneous term}.
“Linear”

Set \( L[y] = y'' + p(x)y' + q(x)y \).

Then, for any two twice differentiable functions \( y_1(x) \) and \( y_2(x) \),

\[
L[y_1(x) + y_2(x)] = L[y_1(x)] + L[y_2(x)]
\]

and, for any constant \( c \),

\[
L[cy(x)] = cL[y(x)].
\]

That is, \( L \) is a linear differential operator.
THEOREM: Given the second order linear equation (1). Let $a$ be any point on the interval $I$, and let $\alpha$ and $\beta$ be any two real numbers. Then the initial-value problem

$$y'' + p(x) y' + q(x) y = f(x),$$

$$y(a) = \alpha, \ y'(a) = \beta$$

has a unique solution.
The linear differential equation

\[ y'' + p(x)y' + q(x)y = f(x) \quad (1) \]

is **homogeneous** if the function \( f \) on the right side is 0 for all \( x \in I \). That is,

\[ y'' + p(x)y' + q(x)y = 0. \]

is a **linear homogeneous** equation.
If $f$ is not the zero function on $I$, that is, if $f(x) \neq 0$ for some $x \in I$, then

$$y'' + p(x)y' + q(x)y = f(x)$$

is a linear nonhomogeneous equation.

**Important Note:** Linear differential equations DO NOT HAVE singular solutions.
Section 3.2. Homogeneous Equations

\[ y'' + p(x) y' + q(x) y = 0 \quad \text{(H)} \]

where \( p \) and \( q \) are continuous functions on some interval \( I \).

The zero function, \( y(x) = 0 \) for all \( x \in I \), \( y \equiv 0 \) is a solution of (H).

The zero solution is called the \textbf{trivial solution}. Any other solution is a \textbf{non-trivial solution}. 

Basic Theorems

**THEOREM 1:** If $y = y_1(x)$ and $y = y_2(x)$ are any two solutions of (H), then

$$u(x) = y_1(x) + y_2(x)$$

is also a solution of (H).

The sum of any two solutions of (H) is also a solution of (H). (Some call this property the *superposition principle*).
Proof:
**THEOREM 2:** If $y = y(x)$ is a solution of (H) and if $C$ is any real number, then

$$u(x) = Cy(x)$$

is also a solution of (H).

Any constant multiple of a solution of (H) is also a solution of (H).
**DEFINITION:** Let \( y = y_1(x) \) and \( y = y_2(x) \) be functions defined on some interval \( I \), and let \( C_1 \) and \( C_2 \) be real numbers. The expression

\[ C_1y_1(x) + C_2y_2(x) \]

is called a **linear combination** of \( y_1 \) and \( y_2 \).
Theorems 1 & 2 can be restated as:

**THEOREM 3:** If $y = y_1(x)$ and $y = y_2(x)$ are any two solutions of (H), and if $C_1$ and $C_2$ are any two real numbers, then

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

is also a solution of (H).

Any linear combination of solutions of (H) is also a solution of (H).
**NOTE:** \( y(x) = C_1 y_1(x) + C_2 y_2x \)

is a two-parameter family which "looks like" the general solution.

Is it???
Some Examples from Chapter 1:

\[ y_1 = \cos 3x \text{ and } y_2 = \sin 3x \]

are solutions of

\[ y'' + 9y = 0 \quad (\text{Exer. 1.3 \#5}) \]

\[ y = C_1 \cos 3x + C_2 \sin 3x \]

is the general solution.
\[ y_1 = x^2 \quad \text{and} \quad y_2 = x^2 \ln x \]

give solutions of
\[ y'' - \frac{3}{x} y' + \frac{4}{x^2} y = 0 \quad (\text{Exer. 1.3 \#6}) \]

\[ y = C_1 x^2 + C_2 x^2 \ln x \]

is the general solution.
Example: \[ y'' - \frac{1}{x}y' - \frac{15}{x^2}y = 0 \]

a. Solutions

\[ y_1(x) = x^5, \quad y_2(x) = 3x^5 \]

General solution:

\[ y = C_1x^5 + C_2(3x^5) \]

That is, is EVERY solution a linear combination of

\[ y_1 \quad \text{and} \quad y_2 \]
Gen. solution: \( y = C_1 x^5 + C_2 (3x^5) \) ??

\( u(x) = x^{-3} \) is a solution
Gen. solution:  \( y = C_1 x^5 + C_2 (3x^5) \) ??

Initial value problems.

\[ y(1) = \quad y'(1) = \]
b. Solutions:

\[ y_1(x) = x^5, \quad y_2(x) = x^{-3} \]

General solution: \( y = C_1 x^5 + C_2 x^{-3} \)

Let \( u = u(x) \) be any solution of the equation
The general case: Let

\[ y = C_1 y_1(x) + C_2 y_2(x) \]

be a two parameter family of solutions of (H). When is this the general solution of (H)?

Answer: When \( y_1 \) and \( y_2 \) ARE NOT CONSTANT MULTIPLES OF EACH OTHER.
**DEFINITION:** Let \( y = y_1(x) \) and \( y = y_2(x) \) be solutions of (H). The function \( W \) defined by

\[
W[y_1, y_2](x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)
\]

is called the **Wronskian** of \( y_1, y_2 \).

**Determinant notation:**

\[
W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}
\]
**THEOREM 4:** Let \( y = y_1(x) \) and \( y = y_2(x) \) be solutions of equation (H), and let \( W(x) \) be their Wronskian. Exactly one of the following holds:

(i) \( W(x) = 0 \) for all \( x \in I \) and \( y_1 \) is a constant multiple of \( y_2 \) AND

\[
y = C_1y_1(x) + C_2y_2(x)
\]

**IS NOT** the general solution of (H)

**OR**
(ii) $W(x) \neq 0$ for all $x \in I$ and

$$y = C_1y_1(x) + C_2y_2(x)$$

IS the general solution of (H)

$W(x)$ is a solution of

$$y' + p(x)y = 0.$$ 

See Section 2.1, Special Case.

The Proof is in the text.
**DEFINITION:** A pair of solutions

\[ y = y_1(x), \quad y = y_2(x) \]

of equation (H) forms a **fundamental set of solutions** (also called a **solution basis**) if

\[ W[y_1, y_2](x) \neq 0 \quad \text{for all} \ x \in I. \]
Section 3.3. Homogeneous Equations with Constant Coefficients

Fact: In contrast to first order linear equations, there are no general methods for solving

\[ y'' + p(x)y' + q(x)y = 0. \quad (H) \]

But, there is a special case of (H) for which there is a solution method, namely
$$y'' + ay' + by = 0 \quad (1)$$

where \( a \) and \( b \) are constants.

**Solutions:** (1) has solutions of the form

$$y = e^{rx}$$
$y = e^{rx}$ is a solution of (1) if and only if

$$r^2 + ar + b = 0$$  \hspace{1cm} (2)

Equation (2) is called the characteristic equation of equation (1)
Note the correspondence:

Diff. Eqn: \[ y'' + ay' + by = 0 \]

Char. Eqn: \[ r^2 + ar + b = 0 \]

The solutions of \[ y'' + ay' + by = 0 \] are determined by the roots of \[ r^2 + ar + b = 0. \]
There are three cases:

1. $r^2 + ar + b = 0$ has two, distinct real roots, $r_1 = \alpha$, $r_2 = \beta$.

2. $r^2 + ar + b = 0$ has only one real root, $r = \alpha$.

3. $r^2 + ar + b = 0$ has complex conjugate roots, $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, $\beta \neq 0$. 
Case I: Two, distinct real roots.

\[ r^2 + ar + b = 0 \] has two distinct real roots:

\[ r_1 = \alpha, \quad r_2 = \beta, \quad \alpha \neq \beta. \]

Then

\[ y_1(x) = e^{\alpha x} \quad \text{and} \quad y_2(x) = e^{\beta x} \]

are solutions of \[ y'' + ay' + by = 0. \]
\[ y_1 = e^{\alpha x} \quad \text{and} \quad y_2 = e^{\beta x} \] are not constant multiples of each other, \( \{y_1, y_2\} \) is a fundamental set,

\[ W[y_1, y_2] = \]

General solution:

\[ y = C_1 e^{\alpha x} + C_2 e^{\beta x} \]
Example 1: Find the general solution of

\[ y'' - 3y' - 10y = 0. \]

(see Example 6, Chapter 1 - Notes)
Example 2: Find the general solution of

\[ y'' - 11y' + 28y = 0. \]
Case II: Exactly one real root.

\[ r = \alpha; \quad (\alpha \text{ is a double root}). \] Then

\[ y_1(x) = e^{\alpha x} \]

is one solution of \( y'' + ay' + by = 0. \)

We need a second solution which is independent of \( y_1. \)
**NOTE:** In this case, the characteristic equation is

\[(r - \alpha)^2 = r^2 - 2\alpha r + \alpha^2 = 0\]

so the differential equation is

\[y'' - 2\alpha y' + \alpha^2 y = 0\]
\[ y = C e^{\alpha x} \] is a solution for any constant \( C \). Replace \( C \) by a function \( u \) which is to be determined so that

\[ y = u(x) e^{\alpha x} \]

is a solution of:

\[ y'' - 2\alpha y' + \alpha^2 y = 0 \]
$y_1 = e^{\alpha x}$ and $y_2 = xe^{\alpha x}$ are not constant multiples of each other, \( \{y_1, y_2\} \) is a fundamental set,

\[ W[y_1, y_2] = \]

General solution:

\[ y = C_1 e^{\alpha x} + C_2 xe^{\alpha x} \]
Examples:

1. Find the general solution of

\[ y'' + 6y' + 9y = 0. \]
2. Find the general solution of

\[ y'' - 10y' + 25y = 0. \]
Case III: Complex conjugate roots.

\[ r_1 = \alpha + i\beta, \quad r_2 = \alpha - i\beta, \quad \beta \neq 0 \]

In this case

\[ u_1(x) = e^{(\alpha+i\beta)x} \quad u_2(x) = e^{(\alpha-i\beta)x} \]

are ind. solns. of \( y'' + ay' + by = 0 \)

and

\[ y = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x} \]

is the general solution. BUT, these are complex-valued functions!! No good!!

We want real-valued solutions!!
Recall from Calculus II:

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \]

\[ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \pm \frac{x^{2n}}{(2n)!} + \cdots \]

\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots \pm \frac{x^{2n-1}}{(2n-1)!} + \cdots \]
\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \]
Relationships between the exponential function, sine and cosine

Euler’s Formula: \( e^{i\theta} = \cos \theta + i \sin \theta \)

These follow:

\( e^{-i\theta} = \cos \theta - i \sin \theta \)

\( \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \)

\( \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \)

\( e^{i\pi} + 1 = 0 \)
\( e^{(\alpha + i \beta)x} = \)
\{u_1 = e^{(\alpha+i\beta)x}, \quad u_2 = e^{(\alpha-i\beta)x}\}

transform into

\{y_1 = e^{\alpha x} \cos \beta x, \quad y_2 = e^{\alpha x} \sin \beta x\}

\ y_1 \ \text{and} \ \ y_2 \ \text{are not constant multiples of each other,} \ \{y_1, y_2\} \ \text{is a fundamental set,} \ W[y_1, y_2] =

\ W[y_1, y_2] = \beta e^{2\alpha x} \neq 0
AND

\[ y = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x \]

is the general solution.
Examples: Find the general solution of

1. \( y'' - 4y' + 13y = 0. \)

2. \( y'' + 6y' + 25y = 0. \)
Comprehensive Examples:

1. Find the general solution of

\[ y'' + 6y' + 8y = 0. \]
2. Find a solution basis for

\[ y'' - 10y' + 25y = 0. \]
3. Find the solution of the initial-value problem

\[ y'' - 4y' + 8y = 0, \quad y(0) = 1, \quad y'(0) = -2. \]
4. Find the differential equation that has

\[ y = C_1 e^{2x} + C_2 e^{-3x} \]

as its general solution. (C.f. Chap 1.)
5. Find the differential equation that has

\[ y = C_1 e^{2x} + C_2 x e^{2x} \]

as its general solution. (C.f. Chap 1.)
6. \( y = 5xe^{-4x} \) is a solution of a second order homogeneous equation with constant coefficients.

a. What is the equation?

b. What is the general solution?
7. $y = 2e^{2x} \sin 4x$ is a solution of a second order homogeneous equation with constant coefficients.

a. What is the equation?

b. What is the general solution?
From Exercises 1.3:

18. \( y = C_1 e^x + C_2 e^{-2x} \).

19. \( y = C_1 e^{2x} + C_2 x e^{2x} \)
22. \[ y = C_1 \cos 3x + C_2 \sin 3x. \]

24. \[ y = C_1 e^{2x} \cos 3x + C_2 e^{2x} \sin 3x. \]