A _second order linear differential equation_ is an equation which can be written in the form

\[ y'' + p(x)y' + q(x)y = f(x) \]

where \( p, q, \) and \( f \) are continuous functions on some interval \( I. \)

The functions \( p \) and \( q \) are called the _coefficients_ of the equation.
The function $f$ is called the **forcing function** or the **nonhomogeneous term**.

\[
y'' + p(x) y' + q(x) y = f(x)
\]
“Linear”

Set \( L[y] = y'' + p(x)y' + q(x)y \). \( \text{eq} \)

\* \( L(c_1 y_1 + c_2 y_2) = c_1 L[y_1] + c_2 L[y_2] \)

Then, for any two twice differentiable functions \( y_1(x) \) and \( y_2(x) \),

\( L[y_1(x) + y_2(x)] = L[y_1(x)] + L[y_2(x)] \)

and, for any constant \( c \),

\( L[cy(x)] = cL[y(x)] \).

That is, \( L \) is a **linear differential operator**.
THEOREM: Given the second order linear equation (1). Let \( a \) be any point on the interval \( I \), and let \( \alpha \) and \( \beta \) be any two real numbers. Then the initial-value problem

\[
\begin{align*}
y'' + p(x) y' + q(x) y &= f(x), \\
y(a) &= \alpha, \quad y'(a) = \beta
\end{align*}
\]

has a unique solution.
The linear differential equation

\[ y'' + p(x)y' + q(x)y = f(x) = 0 \]  \hspace{1cm} (1)

is homogeneous if the function \( f \) on the right side is 0 for all \( x \in I \). That is,

\[ y'' + p(x)y' + q(x)y = 0. \] \hspace{1cm} (H)

is a linear homogeneous equation.
If $f$ is not the zero function on $I$, that is, if $f(x) \neq 0$ for some $x \in I$, then

$$y'' + p(x)y' + q(x)y = f(x)$$

is a **linear nonhomogeneous** equation.

**Important Note:** Linear differential equations DO NOT HAVE singular solutions.
Section 3.2. Homogeneous Equations

\[ y'' + p(x)y' + q(x)y = 0 \quad (H) \]

where \( p \) and \( q \) are continuous functions on some interval \( I \).

The zero function, \( y(x) = 0 \) for all \( x \in I \), \( (y \equiv 0) \) is a solution of (H).

The zero solution is called the trivial solution. Any other solution is a non-trivial solution.
Basic Theorems

**THEOREM 1:** If \( y = y_1(x) \) and \( y = y_2(x) \) are any two solutions of (H), then
\[
u(x) = y_1(x) + y_2(x)
\]
is also a solution of (H).

\[
u'' + p(x)\nu' + q(x)\nu = [y_1'' + y_2'' + p(x)(y_1 + y_2)] + [y_1' + p(x)y_1' + q(x)y_1] + [y_2' + p(x)y_2' + q(x)y_2] = 0 + 0 = 0
\]
The sum of any two solutions of (H) is also a solution of (H). (Some call this property the **superposition principle**).
Proof: \( \text{II. Way} \).

\( y_1 \) and \( y_2 \) are soln of \( (\mathbb{H}) \).

\[ L[y_1] = 0 \quad \text{and} \quad L[y_2] = 0 \]

\[ L[y_1 + y_2] = L[y_1] + L[y_2] \]

\[ = 0 + 0 \]

Thus, \( y_1 + y_2 \) is also a soln of \( (\mathbb{H}) \).
THEOREM 2: If \( y = y(x) \) is a solution of (H) and if \( C \) is any real number, then

\[
u(x) = Cy(x)\]

is also a solution of (H).

Any constant multiple of a solution of (H) is also a solution of (H).

Proof: \( L[y] = 0 \) since \( y \) is a sol. of (H).

Then \( L[Cy] = C \underbrace{L[y]} = C \cdot 0 = 0 \)

\( \Rightarrow Cy \) is a sol. of (H).
**DEFINITION:** Let \( y = y_1(x) \) and \( y = y_2(x) \) be functions defined on some interval \( I \), and let \( C_1 \) and \( C_2 \) be real numbers. The expression \( C_1y_1(x) + C_2y_2(x) \) is called a **linear combination** of \( y_1 \) and \( y_2 \).
Theorems 1 & 2 can be restated as:

**THEOREM 3:** If \( y = y_1(x) \) and \( y = y_2(x) \) are any two solutions of (H), and if \( C_1 \) and \( C_2 \) are any two real numbers, then

\[
y(x) = C_1 y_1(x) + C_2 y_2(x)
\]

is also a solution of (H).

Any linear combination of solutions of (H) is also a solution of (H).
NOTE: \[ y(x) = C_1 y_1(x) + C_2 y_2 x \]

is a two-parameter family which "looks like" the general solution. \[ y'' + p(x) y' + q(x) y = 0 \]

Is it???
Some Examples from Chapter 1:

\[ y_1 = \cos 3x \quad \text{and} \quad y_2 = \sin 3x \]

\[
\begin{align*}
y_1' &= -3\sin 3x \\
y_1'' &= -9\cos 3x \\
y_2' &= 3\cos 3x \\
y_2'' &= -9\sin 3x
\end{align*}
\]

are solutions of

\[
y_1'' + 9y_1 = -9\cos 3x + 9(\cos 3x) = 0 < \text{YEO}
\]

\[ \rightarrow y'' + 9y = 0 \quad \text{(Exer. 1.3 #5)} \]

\[
y_2'' + 9y_2 = -9\sin 3x + 9(\sin 3x) = 0 \quad \text{YES}
\]

\[
y = C_1 \cos 3x + C_2 \sin 3x
\]

\[ y = c_1 y_1 + c_2 y_2 \]

is the general solution.
\[ y_1 = x^2 \quad \text{and} \quad y_2 = x^2 \ln x \]

are solutions of

\[ y'' - \frac{3}{x} y' + \frac{4}{x^2} y = 0 \quad (\text{Exer. 1.3 \#6}) \]

\[ y = C_1 x^2 + C_2 x^2 \ln x \]

is the general solution.
Example: \[ y'' - \frac{1}{x} y' - \frac{15}{x^2} y = 0 \]

a. Solutions

\[ y_1(x) = x^5, \quad y_2(x) = 3x^5 \]

General solution: \[ y = C_1 x^5 + C_2 (3x^5) \] ??

\[ y_2 = 3x^5 = 3y_1 \]

That is, is EVERY solution a linear combination of \( y_1 \) and \( y_2 \)?
Gen. solution: \[ y = C_1 x^5 + C_2 (3x^5) \]

\[ = (C_1 + 3C_2) x^5 = M x^5 \]

\[ u(x) = x^{-3} \text{ is a solution} \]

\[ u'(x) = -3x^{-4} \]
\[ u''(x) = 12x^{-5} = \frac{12}{x^5} \]

Consider this DE set \[ y = x^r \] find \( r \) such that \( y \) is a soln of 1.

\[ y' = rx^{r-1} \]
\[ y'' = r (r-1)x^{r-2} \]

\[ r (r-1)x^{r-2} - \frac{1}{x} rx^{r-1} - \frac{15}{x^2} x^{r-2} = 0 \]

\[ r (r-1)x^{r-2} - 15x^{r-2} = 0 \]

\[ r^2 - r - 15 = 0 \]

\[ r = -3, r = 5 \]

\[ y_1 = x^{-3}, \quad y_2 = x^5 \]
Gen. solution: $y = C_1 x^5 + C_2 (3x^5)$

Initial value problems.

$$y(1) = 11 \quad y'(1) = 3$$

Let $y$ be the solution of (1), such that $y(1) = 11$ and $y'(1) = 3$.

Therefore, $y = C_1 x^5 + C_2 (3x^5)$.

$$y' = 5C_1 x^4 + 15C_2 x^4$$

$$\begin{align*}
y(1) &= C_1 + 3C_2 = 11 \\
y'(1) &= 5C_1 + 15C_2 = 3
\end{align*}$$

Solving the system:

$$0.5C_1 + 0.5C_2 = -52$$

No solutions!
b. Solutions:

\[ y_1(x) = x^5, \quad y_2(x) = x^{-3} \]

Yes.

General solution: \( y = C_1x^5 + C_2x^{-3} \) ?

Suppose \( y(1) = 11 \) \[ y' = 5C_1x^4 - 3C_2x^{-4} \]
\( y'(1) = 3 \)

Let \( u = u(x) \) be any solution of the equation

\[
\begin{align*}
    y(1) &= C_1 + C_2 = 11 \quad \text{(3)} \\
    y'(1) &= 5C_1 - 3C_2 = 3 \\
    -8C_2 &= -52 \Rightarrow C_2 = \frac{52}{8} \\
    8C_1 &= 36 \Rightarrow C_1 = \frac{36}{8}.
\end{align*}
\]

\[ y = \frac{36}{8} x^5 + \frac{52}{8} x^{-5} = \frac{9}{2} x^5 + \frac{13}{2} x^{-3}. \]
The general case: Let

\[ y = C_1 y_1(x) + C_2 y_2(x) \]

be a two parameter family of solutions of (H). When is this the general solution of (H)?

Answer: When \( y_1 \) and \( y_2 \) ARE NOT CONSTANT MULTIPLES OF EACH OTHER.
**DEFINITION:** Let \( y = y_1(x) \) and \( y = y_2(x) \) be solutions of (H). The function \( W \) defined by

\[
W[y_1, y_2](x) = y_1(x)y'_2(x) - y_2(x)y'_1(x)
\]

is called the **Wronskian** of \( y_1, y_2 \).

**Determinant notation:**

\[
W(x) = y_1(x)y'_2(x) - y_2(x)y'_1(x)
\]

\[
W[y_1, y_2] = \begin{vmatrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x) \end{vmatrix}
\]
THEOREM 4: Let \( y = y_1(x) \) and \( y = y_2(x) \) be solutions of equation (H), and let \( W(x) \) be their Wronskian. Exactly one of the following holds:

(i) \( W(x) = 0 \) for all \( x \in I \) and \( y_1 \) is a constant multiple of \( y_2 \) AND

\[
y = C_1 y_1(x) + C_2 y_2(x)
\]

IS NOT the general solution of (H) OR
(ii) \( W(x) \neq 0 \) for all \( x \in I \) and

\[
y = C_1 y_1(x) + C_2 y_2(x)
\]

IS the general solution of (H)

\[
W(x^5, 3x^5) = \begin{vmatrix} x^5 & 3x^5 \\ 5x^4 & 15x^4 \end{vmatrix} = 15x^9 - 15x^9 = 0.
\]

\( W(x) \) is a solution of

\[
y' + p(x)y = 0.
\]

Since \( W[y_1, y_2] = 0 \), \( y = c_1 y_1 + c_2 y_2 \) is not the general solution

See Section 2.1, Special Case.

The Proof is in the text.
Fundamental Set; Solution basis

**DEFINITION:** A pair of solutions

\[ y = y_1(x), \quad y = y_2(x) \]

of equation (H) forms a fundamental set of solutions (also called a solution basis) if

\[ W[y_1, y_2](x) \neq 0 \quad \text{for all} \quad x \in I. \]

\[ y_1 = x^5, \quad y_2 = x^{-3} \]

\[ W(x^5, x^{-3}) = \begin{vmatrix} x^5 & x^{-3} \\ 5x^4 & -3x^{-4} \end{vmatrix} = -3x - 5c = -8x \neq 0 \]
Section 3.3. Homogeneous Equations with Constant Coefficients

Fact: In contrast to first order linear equations, there are no general methods for solving

$$y'' + p(x)y' + q(x)y = 0. \quad (H)$$

But, there is a special case of (H) for which there is a solution method, namely
\[ y'' + ay' + by = 0 \quad (1) \]

where \( a \) and \( b \) are constants.

**Solutions:** (1) has solutions of the form

\[
\begin{align*}
\{ & y'' + ay' + by = 0 \quad \bigcirc \\
& y = e^{rx}, \quad y' = re^{rx}, \quad y'' = r^2 e^{rx} \\
& r^2 e^{rx} + ar e^{rx} + be^{rx} = 0 \\
& e^{rx} [r^2 + ar + b] = 0, \quad e^{rx} \neq 0, \\
& r^2 + ar + b = 0.
\end{align*}
\]
$y = e^{rx}$ is a solution of (1) if and only if

$$y'' + ay' + by = 0$$

$$r^2 + ar + b = 0 \quad (2)$$

Equation (2) is called the **characteristic equation** of equation (1)
Note the correspondence:

Diff. Eqn: \( y'' + ay' + by = 0 \)

\( \Rightarrow \) Char. Eqn: \( r^2 + ar + b = 0 \)

The solutions of

\( y'' + ay' + by = 0 \)

are determined by the roots of

\( r^2 + ar + b = 0. \)
There are three cases:

1. \( r^2 + ar + b = 0 \) has two, distinct real roots, \( r_1 = \alpha, \ r_2 = \beta \).
   \[ r^2 - 5r + 6 = 0, \quad r_1 = 2, \quad r_2 = 3 \]

2. \( r^2 + ar + b = 0 \) has only one real root, \( r = \alpha \).
   \[ r^2 - 4r + 4 = 0 \quad r = 2 \quad \text{double} \]

3. \( r^2 + ar + b = 0 \) has complex conjugate roots, \( r_1 = \alpha + i\beta, \ r_2 = \alpha - i\beta, \ \beta \neq 0 \).
   \[ \alpha, \beta \in \mathbb{R} \quad i^2 = -1 \]
   \[ r^2 + 6r + 13 = 0 \]
   \[ r_1 = -3 + 2i, \quad r_2 = -3 - 2i \]
Case I: Two, distinct real roots.

\[ r^2 + ar + b = 0 \] has two distinct real roots:

\[ r_1 = \alpha, \quad r_2 = \beta, \quad \alpha \neq \beta. \]

Then

\[ y_0 = e^{\alpha x} \]

are solutions of

\[ y'' + ay' + by = 0. \]

\[
W(e^{\alpha x}, e^{\beta x}) = \begin{vmatrix}
\alpha x & \beta x \\
\alpha e & \beta e
\end{vmatrix} = \beta e^{\alpha x + \beta x} - \alpha e^{\alpha x + \beta x} = e^{\alpha x + \beta x} (\beta - \alpha)
\]

\[ W \neq 0, \text{ for } \alpha \neq \beta. \]

The general solution of

\[ y = c_1 e^{\alpha x} + c_2 e^{\beta x}. \]
$y_1 = e^{\alpha x}$ and $y_2 = e^{\beta x}$ are not constant multiples of each other, \( \{y_1, y_2\} \) is a fundamental set,

$$W[y_1, y_2] = \begin{vmatrix} e^{\alpha x} & e^{\beta x} \\ \alpha x & \beta x \end{vmatrix} = (\beta - \alpha)e^{x(\alpha + \beta)} \neq 0$$

General solution:

$$y = C_1 e^{\alpha x} + C_2 e^{\beta x}$$

Case 1: If the roots of characteristic equation are distinct and real then the solution of (4) is

$$y = c_1 e^{\alpha x} + c_2 e^{\beta x}$$
Example 1: Find the general solution of

\[ y'' - 3y' - 10y = 0. \]

(see Example 6, Chapter 1 - Notes)

\[ r^2 - 3r - 10 = 0 \]
\[ (r - 5)(r + 2) = 0 \]
\[ r_1 = 5, \quad r_2 = -2 \]
\[ y_1 = e^{5x}, \quad y_2 = e^{-2x} \]
\[ -2 \pm 5. \]

\[ W(y_1, y_2) \neq 0, \quad \text{Thus} \]
\[ y = c_1 e^{5x} + c_2 e^{-2x} \quad \text{general} \]

Solve DE.

\[ 2y'' - 5y' - 3y = 0 \quad \text{solve DE.} \]

\[ \begin{align*}
2r^2 - 5r - 3 &= 0 \\
(r - 3)(2r + 1) &= 0 \\
r_1 &= -\frac{1}{2}, \quad r_2 = 3
\end{align*} \]

\[ \begin{align*}
\text{distinct and real} \\
y &= c_1 e^{-\frac{1}{2}x} + c_2 e^{3x}
\end{align*} \]
\( y'' - y' - 6y = 0 \)

\( r^2 - r - 6 = 0 \)

\( (r-3)(r+2) = 0 \)

\[ r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

\( a = 1, \ b = -1, \ c = -6. \)

Distinct real roots:

\( r_1 = 3, \ r_2 = -2 \)

General soln:

\[ y = c_1 e^{3x} + c_2 e^{-2x} \]
Example 2: Find the general solution of

\[ sr^2 - 11r + 28 = 0 \]

\[ y'' - 11y' + 28y = 0. \]

Characteristic Eq.

\[(r-7) (r-4) = 0, \quad r_1 = 7, \quad r_2 = 4\]

Real distinct

Solution: \[ y = c_1 e^{7x} + c_2 e^{4x} \]
Case II: Exactly one real root.

\[ r = \alpha; \quad (\alpha \text{ is a double root}). \] Then

\[ y_1(x) = e^{\alpha x} \]

is one solution of \( y'' + ay' + by = 0. \)

\[
\begin{align*}
y &= C_1 y_1 + C_2 y_2, \\
y_1 &= e^{\alpha x}, \quad y_1 + C y_2
\end{align*}
\]

We need a second solution which is independent of \( y_1. \)

If \( r_1 \) and \( r_2 \) are repeated roots,

\[
\text{Solve } \begin{align*} y'' - 10y' + 25y &= 0 \\
\text{find soln of DE.}
\end{align*}
\]

\[
\begin{align*}
\text{characteristic eq: } & r^2 - 10r + 25 = 0 \\
& (r-5)^2 = 0 \\
& r = 5 \text{ double}
\end{align*}
\]

\[
y = C_1 e^{5x} + C_2 xe^{5x}
\]
**NOTE:** In this case, the characteristic equation is

\[
(r - \alpha)^2 = r^2 - 2\alpha r + \alpha^2 = 0
\]

so the differential equation is

\[
y'' - 2\alpha y' + \alpha^2 y = 0
\]
Case II. Double Root \( \alpha \): \( r^2 + br + c = 0 \)
\( r_1 = r_2 = \alpha \quad (r-\alpha)^2 = 0 \)

\[ y = Ce^{\alpha x} \] is a solution for any constant

\( y_1 = e^{\alpha x} \), we still need \( y_2 \). I can't pick \( C \). Replace \( C \) by a function \( u \) which is to be determined so that

\[ y_2 = Ce^{\alpha x} = Cy_1 \Rightarrow w = 0, \quad y_1, y_2 \text{ will not form a basis.} \]

\[ (r-\alpha)^2 = 0 \Rightarrow r^2 - 2\alpha \, r + \alpha^2 = 0 \quad \text{characteristic Eq.} \]

\[ y'' - 2\alpha \, y' + \alpha^2 \, y = 0 \]

is a solution of: \( y'' - 2\alpha \, y' + \alpha^2 \, y = 0 \)

\[
\begin{align*}
y_1 &= e^{\alpha x} \\
y_2 &= u(x)e^{\alpha x}
\end{align*}
\]

\[
\begin{align*}
y'' &= 10 \, y' + 25 \, y = 0 \\
r^2 - 10 \, r + 25 = 0 \Rightarrow (r-5)^2 = 0 \\
y_1 &= e^{\alpha x} \\
y_2 &= u(x)e^{\alpha x} \\
y_2' &= \alpha \, \frac{du}{dx} + u \, e^{\alpha x} \\
y_2'' &= \alpha^2 \, u \, e^{\alpha x} + 2 \, e^{\alpha x} \, u' + u \, e^{\alpha x}
\end{align*}
\]

\[ y'' - 2\alpha \, y' + \alpha^2 \, y = 0 \quad \text{(DE)} \]

\[ \alpha^2 \, u \, e^{\alpha x} + 2 \, e^{\alpha x} \, u' + e^{\alpha x} \, u'' = -7 \alpha \,(\alpha \, e^{\alpha x} + u \, e^{\alpha x}) \]

\[ 2 \, e^{\alpha x} \, u' + e^{\alpha x} \, u'' - 2 \alpha \, u \, e^{\alpha x} = 0 \]

\[ \Rightarrow e^{\alpha x} \,(u'' + 2 \, u' - 2 \alpha \, u') = 0 \quad u'' = 0 \quad \text{or} \quad (2 - 2\alpha)u' = 0 \]

\[ u' = 0 \]
\[ u'' = 0 \]
\[ u' = \alpha \Rightarrow u = \alpha \]

* when the roots are repeated. \( r_1 = r_2 = \alpha \)

The general solution will be

\[ y = c_1 e^{\alpha x} + c_2 xe^{\alpha x} \]
\[ y_1 = e^{\alpha x} \] and \[ y_2 = xe^{\alpha x} \] are not constant multiples of each other, \( \{y_1, y_2\} \) is a fundamental set,

\[
W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{\alpha x} & xe^{\alpha x} \\ \alpha x e^{\alpha x} & e^{\alpha x} + \alpha xe^{\alpha x} \end{vmatrix} = \frac{e^{\alpha x}}{2\alpha x} \left( e^{\alpha x} \frac{\alpha x}{e^{\alpha x}} - xe \right) = e^{\alpha x} \neq 0
\]

\[ y_1 = e^{\alpha x} \neq y_2 = xe^{\alpha x} \text{ form a fundamental set of \textit{sort of the}} \]

General solution:

\[ y = C_1 e^{\alpha x} + C_2 xe^{\alpha x} \]
Examples:

1. Find the general solution of

\[ y'' + 6y' + 9y = 0. \]

\[ r^2 + 6r + 9 = 0 \]

\[ (r + 3)^2 = 0 \Rightarrow r = -3 = \lambda \]

Double root -3x -3x

\[ y = c_1 e^{-3x} + c_2 xe^{-3x} \]
2. Find the general solution of

\[ y'' - 10y' + 25y = 0. \]

\[ \rightarrow \text{Find characteristic eq:} \]

\[ r^2 - 10r + 25 = 0 \]

\[ (r-5)^2 = 0 \Rightarrow r = 5 \quad \text{double root} \]

\[ y = c_1 e^{5x} + c_2 xe^{5x} \]

\underline{Ex3} Solve \[ y'' - 14y' + 49y = 0 \]

\[ r^2 - 14r + 49 = 0. \]

\[ (r-7)^2 = 0 \Rightarrow r = 7 \quad \text{double root} \]

\[ y = c_1 e^{7x} + c_2 xe^{7x} \]
Case III: Complex conjugate roots.

\[ r_1 = \alpha + i \beta, \quad r_2 = \alpha - i \beta, \quad \beta \neq 0 \]

In this case

\[ u_1(x) = e^{(\alpha + i\beta)x} \quad u_2(x) = e^{(\alpha - i\beta)x} \]

are ind. solns. of \( y'' + ay' + by = 0 \) and

\[ y = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x} \]

is the general solution. BUT, these are complex-valued functions!! No good!! We want real-valued solutions!!
Recall from Calculus II:

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \]

\[ y = c_1 e^{(x + i\beta)x} + c_2 e^{(x - i\beta)x} \text{ instead} \]

\[ y = e^{\alpha x} \left[ c_1 \cos \beta x + c_2 \sin \beta x \right] \]

\[ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \pm \frac{x^{2n}}{(2n)!} + \cdots \]

\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots \pm \frac{x^{2n-1}}{(2n-1)!} + \cdots \]
\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \]

\[ e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots + \frac{(i\theta)^n}{n!} + \cdots \]

\[ = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots \]

\[ = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right) \]

\[ e^{i\theta} = \cos \theta + i \sin \theta \quad \text{Euler's Formula} \]

\[ \theta = \pi, \quad e^{i\pi} = \cos \pi + i \sin \pi = -1 + 0 = -1 \]

\[ e^{i\pi} = -1 \Rightarrow e^{i\pi} + 1 = 0 \]
Relationships between the exponential function, sine and cosine

**Euler’s Formula:** \( e^{i\theta} = \cos \theta + i \sin \theta \)

These follow:

\[
\begin{align*}
e^{i\theta} &= \cos \theta + i \sin \theta \\
e^{-i\theta} &= \cos \theta - i \sin \theta \\
e^{i\theta} + e^{-i\theta} &= 2 \cos \theta \\
2i \sin \theta &= e^{i\theta} - e^{-i\theta} \\
\sin \theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i} \\
e^{i\pi} + 1 &= 0
\end{align*}
\]

\[
\begin{align*}
\sin(-\theta) &= -\sin \theta \\
\cos(-\theta) &= \cos \theta
\end{align*}
\]
\[ y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} \]

Try to get rid of imaginary terms.

\[ y_1 = e^{(\alpha + i\beta)x} = e^{\alpha x} e^{i\beta x} = e^{\alpha x}[\cos \beta x + i \sin \beta x] \]

\[ = e^{\alpha x} \cos \beta x + ie^{\alpha x} \sin \beta x \]

\[ y_2 = e^{(\alpha - i\beta)x} = e^{\alpha x} e^{-i\beta x} = e^{\alpha x}[\cos \beta x - i \sin \beta x] \]

\[ = e^{\alpha x} \cos \beta x - ie^{\alpha x} \sin \beta x \]

\[ \frac{y_1 + y_2}{2} = e^{\alpha x} \cos \beta x, \text{ a real valued soln} \]

\[ \frac{y_1 - y_2}{2i} = e^{\alpha x} \sin \beta x \]
\{ y_1 = e^{(\alpha+i\beta)x}, \quad y_2 = e^{(\alpha-i\beta)x} \}

transform into

\{ y_1 = e^{\alpha x} \cos \beta x, \quad y_2 = e^{\alpha x} \sin \beta x \}

\( y_1 \) and \( y_2 \) are not constant multiples of each other, \( \{ y_1, y_2 \} \) is a fundamental set, \( W[y_1, y_2] = \)

\[
\begin{vmatrix}
 e^{\alpha x} \cos \beta x & e^{\alpha x} \sin \beta x \\
 e^{\alpha x} \cos \beta x - e^{2\alpha x} \sin \beta x & e^{\alpha x} \sin \beta x + e^{2\alpha x} \cos \beta x
\end{vmatrix}
\]

\[= \beta e^{2\alpha x} \neq 0, \quad W \neq 0.\]
AND

\[ y = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x \]

is the general solution.

when the roots of charac. eq are complex conjugate:

\[ \alpha \neq \beta, \beta \neq 0 \]

\[ y = e^{\alpha x} \left[ C_1 \cos \beta x + C_2 \sin \beta x \right] \]
Examples: Find the general solution of

1. \(y'' - 4y' + 13y = 0.\)
   
   \[r^2 - 4r + 13 = 0, \quad r = \frac{4 \pm \sqrt{16 - 4 \cdot 13}}{2}\]
   
   \[r = 2 + 3i, \quad \alpha = 2, \quad \beta = 3\]
   
   \[y = e^{2x} [c_1 \cos 3x + c_2 \sin 3x]\]

2. \(y'' + 6y' + 25y = 0.\)
   
   \[r^2 + 6r + 25 = 0\]

   \[r = \frac{-6 \pm \sqrt{36 - 4 \cdot 25}}{2} = \frac{-6 \pm 8i}{2}\]
   
   \[r = -3 + 4i, \quad \alpha = -3, \quad \beta = 4\]
   
   \[y = e^{-3x} [c_1 \cos 4x + c_2 \sin 4x]\]
Comprehensive Examples:

1. Find the general solution of

\[ r^2 + 6r + 8 = 0 \]

\[ y'' + 6y' + 8y = 0. \]

\((r + 2)(r + 4) = 0\), \(r = -2, r = -4\) real distinct.

\[ y = c_1 e^{-2x} + c_2 e^{-4x} \]

**Ex** Find the general soln of \( y'' + 9y = 0 \)

\[ r^2 + 9 = 0 \Rightarrow r = \pm 3i \]

\(a = 0, \ b = 3\)

\[ y = e^{0x} \left[ c_1 \cos 3x + c_2 \sin 3x \right] \]

\[ y = c_1 \cos 3x + c_2 \sin 3x \]
2. Find a solution basis for

\[ \frac{d^2y}{dx^2} - 10 \frac{dy}{dx} + 25y = 0. \]

\[ r^2 - 10r + 25 = 0 \]

\[ (r - 5)^2 = 0, \quad r = 5 \text{ double root} \]

\[ y = c_1 e^{5x} + c_2 xe^{5x} \]
3. Find the solution of the initial-value problem

\[ y'' - 4y' + 8y = 0, \quad y(0) = 1, \quad y'(0) = -2. \]

\[ r^2 - 4r + 8 = 0 \]
\[ r = \frac{4 \pm \sqrt{16 - 4 \cdot 8}}{2} = \frac{4 \pm 4i}{2} = 2 \pm 2i \]
\[ a = 2, \quad \beta = 2 \]
\[ y = e^{2x} \left[ c_1 \cos 2x + c_2 \sin 2x \right] \text{ the general solution.} \]

\[ y(0) = 1 \Rightarrow 1 = c_1 \]

\[ y' = 2e^{2x} \left[ c_1 \cos 2x + c_2 \sin 2x \right] + e^{2x} \left[ -2c_1 \sin 2x + 2c_2 \cos 2x \right] \]
\[ -2 = 2c_1 + 2c_2 \Rightarrow -1 = c_1 + c_2, \quad \text{already } c_1 = 1 \]
\[ c_2 = -2 \]

\[ y = e^{2x} \left[ \cos 2x - 2 \sin 2x \right]. \]
4. Find the differential equation that has

\[ y = C_1e^{2x} + C_2e^{-3x} \]

as its general solution. (C.f. Chap 1.)

\( r_1 = 2, \quad r_2 = -3 \) real distinct.

\((r-2)(r+3) = r^2 + r - 6 = 0 \) char.

\[ r^2 + r - 6 = 0 \]

\[ y'' + y' - 6y = 0 \]
5. Find the differential equation that has
\[ y = C_1 e^{2x} + C_2 x e^{2x} \]
as its general solution. (C.f. Chap 1.)

\[(r - 2)^2 = 0\]
\[r^2 - 4r + 4 = 0\] Chor. Eq.
\[y'' - 4y' + 4y = 0.\] DE

If roots \(r_1, r_2\) are real & distinct:
\[y = c_1 e^{r_1 x} + c_2 e^{r_2 x}\]

If roots \(r\) double & real
\[y = c_1 e^{rx} + c_2 xe^{rx}\]

If roots are complex conjugate
\[r = \alpha \pm i \beta\]
\[y = e^{\alpha x} \left[ c_1 \cos \beta x + c_2 \sin \beta x \right]\]
6. \( y = 5xe^{-4x} \) is a solution of a second order homogeneous equation with constant coefficients.

a. What is the equation?

double root \( = -4 \) characteristic eq:

\[
(r + 4)^2 = 0
\]

\[
 r^2 + 8r + 16 = 0
\]

\[
y'' + 8y' + 16y = 0 \quad \Rightarrow \quad \boxed{\text{AE}}
\]

b. What is the general solution?

\[
y = c_1e^{-4x} + c_2xe^{-4x}
\]
7. \( y = 2e^{2x} \sin 4x \) is a solution of a second order homogeneous equation with constant coefficients.

a. What is the equation? 
\[
(r-(2+4i))(r-(2-4i)) = 0
\]
\[
r^2 - 4r + 20 = 0.
\]

b. What is the general solution? 
\[
y = e^{2x} \left[ c_1 \cos 4x + c_2 \sin 4x \right].
\]
From Exercises 1.3:

18. \( y = C_1 e^x + C_2 e^{-2x} \).

\[(r-1)(r+2) = 0\]
\[r^2 + r - 2 = 0\]
\[y'' + y' - 2y = 0.\]

19. \( y = C_1 e^{2x} + C_2 xe^{2x} \) double \( r = 2 \).

\[(r-2)^2 = 0\] ch. eq.
\[r^2 - 4r + 4 = 0\]
\[y''' - 4y' + 4y = 0.\]
22. \[ y = C_1 \cos 3x + C_2 \sin 3x. \]

\[ \gamma = 0 + i\beta = 0 + 3i, \]

\[ (\gamma - 3i)(\gamma + 3i) = 0 \]

\[ r^2 - 3ir - 3ir + 9i^2 = 0 \]

\[ r^2 - 9 = 0 \]

\[ y'' - 9y = 0. \]

\[ \alpha = 2 \quad \beta = 3 \]

24. \[ y = C_1 e^{2x} \cos 3x + C_2 e^{2x} \sin 3x. \]

\[ (\gamma - (2+3i))(\gamma + (2-3i)) = 0 \]

\[ = r^2 - 4r + 13 = 0 \]

\[ y'' - 4y' + 13y = 0. \]