Chapter 2

FIRST ORDER EQUATIONS

\[ F(x, y, y') = 0 \]

**Basic assumption:** The equation can be solved for \( y' \); that is, the equation can be written in the form

\[ y' = f(x, y) \quad (1) \]
2.1. First Order Linear Equations

Equation (1) is a **linear equation** if $f$ has the form

$$f(x, y) = P(x)y + q(x)$$

where $P$ and $q$ are continuous functions on some interval $I$. Thus

$$y' = P(x)y + q(x)$$
Standard form:

The **standard form** for a first order linear equation is:

\[ y' + p(x)y = q(x) \]

where \( p \) and \( q \) are continuous functions on the interval \( I \).
Examples:

1. Find the general solution:

\[ y' = 2y \]
2. Find the general solution:

\[ y' + 2x y = 4x \]
Solution Method:

**Step 1.** Establish that the equation is linear by writing it in standard form

\[ y' + p(x)y = q(x). \]

**Step 2.** Multiply by \( e^{\int p(x) \, dx} \):

\[ e^{\int p(x) \, dx} y' + p(x)e^{\int p(x) \, dx} y = q(x)e^{\int p(x) \, dx} \]

\[ \left[ e^{\int p(x) \, dx} y \right]' = q(x)e^{\int p(x) \, dx} \]
\[
\left[ e^{\int p(x) \, dx} \, y \right]' = q(x) e^{\int p(x) \, dx}
\]

**Step 3.** Integrate:

\[
e^{\int p(x) \, dx} \, y = \int q(x) e^{\int p(x) \, dx} \, dx + C.
\]

**Step 4.** Solve for \( y \):

\[
y = e^{-\int p(x) \, dx} \int q(x) e^{\int p(x) \, dx} \, dx + C e^{-\int p(x) \, dx}.
\]
\[ y = e^{-\int p(x) \, dx} \int q(x)e^{\int p(x) \, dx} \, dx + Ce^{-\int p(x) \, dx}. \]

is the general solution of the equation.

**Note:** \( e^{\int p(x) \, dx} \) is called an integrating factor.
3. Find the general solution:

\[ xy' + 3y = \frac{\ln x}{x^2} \]

\[ y = \frac{\ln x}{x^2} - \frac{1}{x^2} + \frac{C}{x^3} \]
4. Find the general solution:

\[ xy' + 2y = \frac{2}{\sqrt{x^2 - 1}} - 2x^2 \]
5. Solve the initial-value problem:

\[ y' + (\cot x)y = 2 \cos x, \quad y(\pi/2) = 3 \]

\[ y = \frac{5 - \cos 2x}{2 \sin x} \]
6. Find the general solution:

\[ y' + 2x y = 2 \tan x \]

**Answer:**

\[ y = e^{-x^2} \int 2e^{x^2} \tan x \, dx + Ce^{-x^2} \]
The term “linear:”

\[ L[y] = y' + p(x) y \]

is a linear operator:
Examples: \( y' + \frac{2}{x} y = 4x \)

\[
L[y] = y' + \frac{2}{x} y
\]

\[
L[2x^3 + x] = ?
\]

\[
L[e^{3x}] = ?
\]

\[
L[x^2] = ?
\]
Set \( L[y] = y' + p(x)y \)

\[
L[y_1 + y_2] = (y_1 + y_2)' + p(y_1 + y_2)
\]

\[
= y_1' + y_2' + py_1 + py_2
\]

\[
= y_1' + py_1 + y_2' + py_2 = L[y_1] + L[y_2]
\]

\[
L[cy] = (cy)' + p(cy) = cy' + cpy
\]

\[
= c(y' + py) = cL[y]
\]
Any "operation" $L$ that has the two properties:

$$L[y_1 + y_2] = L[y_1] + L[y_2]$$

$$L[cy] = cL[y], \quad c \text{ constant}$$

is a **linear operation**.

1. **Differentiation** is a linear operation

2. **Integration** is a linear operation
3. \( L[y] = y + p(x)y \) is a linear operation.

4. \( L[y] = y'' + p(x)y' + q(x)y \) is a linear operation (Chapter 3).
2.2. Separable Equations

\[ y' = f(x, y) \]

is a **separable equation** if \( f \) has the **factored form**

\[ f(x, y) = p(x)h(y) \]

where \( p \) and \( h \) are continuous functions. Thus

\[ y' = p(x)h(y) \]

is the "standard form" of a separable equation.
Examples:

1. Find the general solution of

\[ y' = 2y \]
Find the general solution of:

\[ y' = xy^2 + x \]
Solution Method

Step 1. Establish that the equation is separable.

Step 2. Divide both sides by \( h(y) \) to “separate” the variables.

\[
\frac{1}{h(y)} y' = p(x) \quad \text{or} \quad q(y) y' = p(x)
\]

which, in differential form, is:

\[
q(y) \, dy = p(x) \, dx.
\]

the variables are “separated.”
Step 3. Integrate

\[ \int q(y) \, dy = \int p(x) \, dx + C \]

\[ Q(y) = P(x) + C \]

where \( Q'(y) = q(y), \quad P'(x) = p(x) \)
Notes:

1. $Q(y) = P(x) + C$ is the general solution. Typically, this is an implicit relation; you may or may not be able to solve it for $y$.

2. $h(y) = 0$ is a source of singular solutions: If $k$ is a number such that $h(k) = 0$, then $y = k$ might be a singular solution.
Examples:

3. Find the general solution:

\[ y' = \frac{2xy^2 + 8x}{4y} \]
4 a. Find the general solution:

\[
\frac{1}{x} \frac{dy}{dx} = e^x \sqrt{y + 1}
\]
4 b. Solve the initial-value problem

\[ \frac{1}{x} \frac{dy}{dx} = e^x \sqrt{y + 1}, \ y(0) = 3 \]
5. Find the general solution:

\[ \frac{dy}{dx} - xy^2 = -x \]
6. Find the general solution:

\[ \frac{dy}{dx} = \frac{e^{x-y}}{1 + e^x} \]
7. Find the general solution and any singular solutions:

\[(1 + x^2 + y^2 + x^2y^2) \frac{dy}{dx} = 2xy^2\]
2.3. Related Equations & Transformations

A. Bernoulli equations

An equation of the form

\[ y' + p(x)y = q(x)y^k, \quad k \neq 0, 1 \]

is called a Bernoulli equation.
The change of variable

\[ v = y^{1-k} \]
	ransforms a Bernoulli equation into

\[ v' + (1 - k)p(x)v = (1 - k)q(x). \]

which has the form

\[ v' + P(x)v = Q(x), \]

a linear equation.
Examples:

1. Find the general solution:

\[ y' - 4y = 2e^x \sqrt{y} \]
2. Find the general solution:

\[ xy' + y = 3x^3 y^2 \]
B. Homogeneous equations

\[ y' = f(x, y) \quad (1) \]

is a **homogeneous equation** if

\[ f(tx, ty) = f(x, y) \]
If (1) is homogeneous, then the change of dependent variable

\[ y = vx, \quad y' = v + xv' \]

transforms (1) into a separable equation:

\[ y' = f(x, y) \rightarrow v + xv' = f(x, vx) = f(1, v) \]

which can be written

\[
\frac{1}{f(1, v) - v} \, dv = \frac{1}{x} \, dx;
\]

the variables are separated.
Examples:

1. Find the general solution:

\[ y' = \frac{x^2 + y^2}{2xy} \]
2. Find the general solution:

\[
\frac{dy}{dx} = \frac{x^2 e^{y/x} + y^2}{xy}
\]
3. Find the general solution:

\[ \frac{dy}{dx} = \frac{y^2}{xy + y^2} \]
2.4. Applications

Orthogonal trajectories

Exponential Growth/Decay

Newton’s Law of Cooling/Heating

Limited Growth (Logistic Equation)

Miscellaneous Models
2.4.1. Orthogonal Trajectories

Example: Family of circles, center at (1, 2):

\[(x - 1)^2 + (y - 2)^2 = C\]

DE for the family:

\[y' = -\frac{x - 1}{y - 2}\]
Circles
Family of lines through \((1, 2)\):

\[ y - 2 = K(x - 1) \]

DE for the family:

\[ y' = \frac{y - 2}{x - 1} \]
Lines
\[ y' = \frac{x - 1}{y - 2} \]

\[ y' = \frac{y - 2}{x - 1} \]

negative reciprocals!!
Lines and circles
Given a one-parameter family of curves

\[ F(x, y, C) = 0. \]

A curve that intersects each member of the family at right angles (orthogonally) is called an orthogonal trajectory of the family.
If
\[ F(x, y, C) = 0 \quad \text{and} \quad G(x, y, K) = 0 \]
are one-parameter families of curves such that each member of one family is an orthogonal trajectory of the other family, then the two families are said to be \textbf{orthogonal trajectories}.
A **procedure** for finding a family of orthogonal trajectories

\[ G(x, y, K) = 0 \]

for a given family of curves

\[ F(x, y, C) = 0 \]

**Step 1.** Determine the differential equation for the given family

\[ F(x, y, C) = 0. \]
Step 2. Replace $y'$ in that equation by $-1/y'$; the resulting equation is the differential equation for the family of orthogonal trajectories.

Step 3. Find the general solution of the new differential equation. This is the family of orthogonal trajectories.
Examples

1. Find the family of orthogonal trajectories of:

\[ y^3 = Cx^2 + 2 \]
\[ y^3 = -x^2 + 2 \]
Orthogonal trajectories:

\[ 3x^2 + 2y^2 + \frac{8}{y} = C \]
Together:
2. Find the orthogonal trajectories of the family of parabolas with vertical axis and vertex at the point \((-1, 3)\).

Eqn. for the family:
Orthogonal trajectories:

\[ \frac{1}{2} (x + 1)^2 + (y - 3)^2 = C \]
Parabolas and ellipses
2.4.2. Radioactive Decay/Exponential Growth

Radioactive Decay

"Experiment:" The rate of decay of a radioactive material at time $t$ is proportional to the amount of material present at time $t$.

Let $A = A(t)$ be the amount of radioactive material present at time $t$. 
Mathematical Model

\[ \frac{dA}{dt} = -r \, A, \quad r > 0 \quad \text{constant}, \]

\[ A(0) = A_0, \quad \text{the initial amount}. \]

\( r \) is the decay rate.

Solution: \( A(t) = A_0 \, e^{-rt} \).

Half-life: \( T = \frac{\ln 2}{r} \).
Example: A certain radioactive material is decaying at a rate proportional to the amount present. If a sample of 50 grams of the material was present initially and after 2 hours the sample lost 10% of its mass, find:
1. An expression for the mass of the material remaining at any time $t$.

2. The mass of the material after 4 hours.
3. How long will it take for 75% of the material to decay?

\[ t \approx 26.32 \text{ hours} \]

4. The half-life of the material.

\[ T \approx 13.16 \text{ hours} \]
Exponential Growth

“Experiment:” Under “ideal” conditions, the rate of increase of a population at time $t$ is proportional to the size of the population at time $t$. Let $P = P(t)$ be the size of the population at time $t$. 
Mathematical Model

\[ \frac{dP}{dt} = kP, \quad k > 0 \text{ constant.} \]

\[ P(0) = P_0, \text{ the initial population.} \]

\( k \) is the **growth rate**.

**Solution:** \( P(t) = P_0 e^{kt}. \)

**Doubling time:** \( T = \frac{\ln 2}{k}. \)
Example: In 1980 the world population was approximately 4.5 billion and in the year 2000 it was approximately 6 billion. Assume that the population increases at a rate proportional to the size of population.
1. Find the growth constant and give the world population at any time $t$.

**Answer:** $P(t) = 4.5 \left( \frac{4}{3} \right)^{t/20}$
2. How long will it take for the world population to reach 9 billion (double the 1980 population)?

**Answer:** $T \approx 48.19$ years (doubling time)
3. The world population for 2016 was reported to be about 7.4 billion. What population does the formula in (1) predict for the year 2016?

Answer: \( P(36) \approx 7.55 \)
Example: It is estimated that the arable land on earth can support a maximum of 30 billion people. Estimate the year when the food supply becomes insufficient to support the world population.

2112
2.4.3. Newton’s Law of Cooling

“Experiment:” The rate of change of the temperature of an object at time $t$ is proportional to the difference between the temperature of the object $u = u(t)$ and the (constant) temperature $\sigma$ of the surrounding medium (e.g., air or water)
Mathematical Model

\[
\frac{du}{dt} = -k(u - \sigma), \quad k > 0 \text{ constant,}
\]
\[
\begin{align*}
  u(0) &= u_0, \quad \text{the initial temperature}.
\end{align*}
\]

Solution:

\[
  u(t) = \sigma + [u_0 - \sigma]e^{-kt}
\]
**Example:** Suppose that a corpse is discovered at 10 p.m. and its temperature is determined to be $85^\circ F$. Two hours later, its temperature is $74^\circ F$. If the ambient temperature is $68^\circ F$, estimate the time of death.

8:52 pm
2.4.6. “Limited” Growth – the Logistic Equation

“Experiment:” Given a population of size $M$. The spread of an infectious disease at time $t$ (or information, or ...) is proportional to the product of the number of people who have the disease $P(t)$ and the number of people who do not $M - P(t)$. 
Mathematical Model:

\[
\frac{dP}{dt} = kP(M - P), \quad k > 0 \text{ constant,}
\]

\[P(0) = R \quad \text{(the number of people who have the disease initially)}\]

Solution: The differential equation is both separable and Bernoulli.

\[P(t) = \frac{MR}{R + (M - R)e^{-Mt}}\]
1. A 1000-gallon cylindrical tank, initially full of water, develops a leak at the bottom. Suppose that the water drains off a rate proportional to the product of the time elapsed and the amount of water present. Let $A(t)$ be the amount of water in the tank at time $t$. 


a. Give the mathematical model (initial-value problem) which describes the process.

\[ \frac{dA}{dt} = ktA, \quad k < 0, \quad A(0) = 1000 \]
b. Find the solution.

\[ A(t) = 1000e^{kt^2/2}. \]
c. Given that 200 gallons of water leak out in the first 10 minutes, find the amount of water, \( A(t) \), left in the tank \( t \) minutes after the leak develops.

\[
A(t) = 1000 \left( \frac{4}{5} \right)^{t^2/100}.
\]
2. A 1000-gallon conical tank, initially full of water, develops a leak at the bottom. Suppose that the water drains off a rate proportional to the product of the time elapsed and the square root of the amount of water present. Let $A(t)$ be the amount of water in the tank at time $t$. 
a. Give the mathematical model (initial-value problem) which describes the process.

\[
\frac{dA}{dt} = kt\sqrt{A}, \quad k < 0, \quad A(0) = 1000
\]
b. Find the solution

\[ A(t) = \left( \frac{1}{4}kt^2 + 10\sqrt{10} \right)^2. \]
3. A disease is spreading through a city of population $M$ (constant). Let $P(t)$ be the number of people who have the disease at time $t$. Suppose that $R$ people had the disease initially and that the rate at which the disease is spreading at time $t$ is proportional to product of the time elapsed and the number of people who don’t have the disease.
a. Give the mathematical model (initial-value problem) which describes the process.

\[ \frac{dP}{dt} = kt(M - P), \quad P(0) = R \]
b. Find the solution.

\[ P(t) = M - (M - R)e^{-kt^2/2} \]
c Find \( \lim_{t \to \infty} P(t) \) and interpret the result.

\[
\lim_{t \to \infty} P(t) = M; \quad \text{everyone gets the disease.}
\]
Existence and Uniqueness Theorem

Given the initial-value problem

\[ y' = f(x, y) \quad y(a) = b. \]

If \( f \) and \( \frac{\partial f}{\partial y} \) are continuous on a rectangle

\[ R: a - \alpha \leq x \leq a + \alpha, \quad b - \beta \leq y \leq b + \beta, \quad \alpha, \beta > 0, \]

then there is an interval

\[ a - h \leq x \leq a + h, \quad h \leq \alpha \]

on which the initial-value problem has a unique solution \( y = y(x) \).