Solving First Order PDEs

Atife Caglar

University of Houston

Partial Differential Equations
Lecture 2
Solving the transport equation

**Goal:** Determine every function $u(x, t)$ that solves

$$\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} = 0,$$

where $v$ is a fixed constant.

**Idea:** Perform a *linear change of variables* to eliminate one partial derivative:

$$\alpha = ax + bt,$$
$$\beta = cx + dt,$$

where:

- $x, t$: original independent variables,
- $\alpha, \beta$: new independent variables,
- $a, b, c, d$: constants to be chosen “conveniently,”
  must satisfy $ad - bc \neq 0$. 

Caglar
First Order PDEs
We use the *multivariable chain rule* to convert to $\alpha$ and $\beta$ derivatives:

\[
\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \alpha} \frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta} \frac{\partial \beta}{\partial x} = a \frac{\partial u}{\partial \alpha} + c \frac{\partial u}{\partial \beta},
\]

\[
\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \alpha} \frac{\partial \alpha}{\partial t} + \frac{\partial u}{\partial \beta} \frac{\partial \beta}{\partial t} = b \frac{\partial u}{\partial \alpha} + d \frac{\partial u}{\partial \beta}.
\]

Hence

\[
\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} = \left( b \frac{\partial u}{\partial \alpha} + d \frac{\partial u}{\partial \beta} \right) + v \left( a \frac{\partial u}{\partial \alpha} + c \frac{\partial u}{\partial \beta} \right)
\]

\[
= (b + av) \frac{\partial u}{\partial \alpha} + (d + cv) \frac{\partial u}{\partial \beta}.
\]
Choosing \( a = 0, b = 1, c = 1, d = -\nu \), the original PDE becomes

\[
\frac{\partial u}{\partial \alpha} = 0.
\]

This tells us that

\[
u = f(\beta) = f(cx + dt) = f(x - \nu t)
\]

for any (differentiable) function \( f \).

**Theorem**

The general solution to the transport equation

\[
\frac{\partial u}{\partial t} + \nu \frac{\partial u}{\partial x} = 0
\]

is given by

\[
u(x, t) = f(x - \nu t),
\]

where \( f \) is any differentiable function of one variable.
Example

Solve the transport equation \( \frac{\partial u}{\partial t} + 3 \frac{\partial u}{\partial x} = 0 \) given the initial condition

\[ u(x, 0) = xe^{-x^2}, \quad -\infty < x < \infty. \]

Solution: We know that the general solution is given by

\[ u(x, t) = f(x - 3t). \]

To find \( f \) we use the initial condition:

\[ f(x) = f(x - 3 \cdot 0) = u(x, 0) = xe^{-x^2}. \]

Thus

\[ u(x, t) = (x - 3t)e^{-(x-3t)^2}. \]
Interpreting the solutions of the transport equation

In three dimensions (xtu-space):
- The graph of the solution is the surface obtained by translating $u = f(x)$ along the vector $\mathbf{v} = \langle v, 1 \rangle$;
- The solution is constant along lines (in the xt-plane) parallel to $\mathbf{v}$.
If we plot the solution $u(x, t) = f(x - vt)$ in the $xu$-plane, and animate $t$:

- $f(x) = u(x, 0)$ is the *initial condition* (concentration);
- $u(x, t)$ is a *traveling wave* with velocity $v$ and shape given by $u = f(x)$.
**In general:** a linear change of variables can always be used to convert a PDE of the form

\[ A \frac{\partial u}{\partial x} + B \frac{\partial u}{\partial y} = C(x, y, u) \]

into an “ODE,” i.e. a PDE containing only one partial derivative.

**Example**

**Solve** \[ 5 \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = x \] **given the initial condition**

\[ u(x, 0) = \sin 2\pi x, \quad -\infty < x < \infty. \]

**Solution:** As above, we perform the linear change of variables

\[ \alpha = ax + bt, \]
\[ \beta = cx + dt. \]
We find that

\[ 5 \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 5 \left( b \frac{\partial u}{\partial \alpha} + d \frac{\partial u}{\partial \beta} \right) + \left( a \frac{\partial u}{\partial \alpha} + c \frac{\partial u}{\partial \beta} \right) \]

\[ = (a + 5b) \frac{\partial u}{\partial \alpha} + (c + 5d) \frac{\partial u}{\partial \beta}. \]

We choose \( a = 1, \quad b = 0, \quad c = 5, \quad d = -1. \) Note that

\[ ad - bc = -1 \neq 0, \]

\[ \alpha = ax + bt = x, \]

\[ \beta = cx + dt = 5x - t. \]

So the PDE (in the variables \( \alpha, \beta \)) becomes

\[ \frac{\partial u}{\partial \alpha} = \alpha. \]
Integrating with respect to $\alpha$ yields

$$u = \frac{\alpha^2}{2} + f(\beta) = \frac{x^2}{2} + f(5x - t).$$

The initial condition tells us that

$$\frac{x^2}{2} + f(5x) = u(x, 0) = \sin 2\pi x.$$ 

If we replace $x$ with $x/5$, we get

$$f(x) = \sin \frac{2\pi x}{5} - \frac{x^2}{50}.$$
Therefore

\[ u(x, t) = \frac{x^2}{2} + f(5x - t) \]

\[ = \frac{x^2}{2} + \sin \left( \frac{2\pi(5x - t)}{5} - \frac{(5x - t)^2}{50} \right) \]

\[ = \frac{xt}{5} - \frac{t^2}{50} + \sin \left( \frac{2\pi(5x - t)}{5} \right). \]

**Remark:** There are an infinite number of choices for \( a, b, c, d \) that will “correctly” eliminate either \( \alpha \) or \( \beta \) from the PDE. Although they may appear different, the solutions obtained are always independent of the choice made.
**Goal:** Develop a technique to solve the (somewhat more general) first order PDE

\[
\frac{\partial u}{\partial x} + p(x, y) \frac{\partial u}{\partial y} = 0. \tag{1}
\]

**Idea:** Look for *characteristic curves* in the $xy$-plane along which the solution $u$ satisfies an ODE.

Consider $u$ along a curve $y = y(x)$. On this curve we have

\[
\frac{d}{dx} u(x, y(x)) = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{dy}{dx}. \tag{2}
\]
Comparing (1) and (2), if we require

\[ \frac{dy}{dx} = p(x, y), \quad (3) \]

then the PDE becomes the ODE

\[ \frac{d}{dx} u(x, y(x)) = 0. \quad (4) \]

These are the characteristic ODEs of the original PDE.
If we express the general solution to (3) in the form \( \varphi(x, y) = C \), each value of \( C \) gives a characteristic curve.
Equation (4) says that \( u \) is constant along the characteristic curves, so that

\[ u(x, y) = f(C) = f(\varphi(x, y)). \]
The Method of Characteristics - Special Case

Summarizing the above we have:

Theorem

The general solution to

\[ \frac{\partial u}{\partial x} + p(x, y) \frac{\partial u}{\partial y} = 0 \]

is given by

\[ u(x, y) = f(\varphi(x, y)), \]

where:

- \( \varphi(x, y) = C \) gives the general solution to \( \frac{dy}{dx} = p(x, y) \), and
- \( f \) is any differentiable function of one variable.
Example

Solve \(2y \frac{\partial u}{\partial x} + (3x^2 - 1) \frac{\partial u}{\partial y} = 0\) by the method of characteristics.

Solution: We first divide the PDE by \(2y\) obtaining

\[
\frac{\partial u}{\partial x} + \frac{3x^2 - 1}{2y} \frac{\partial u}{\partial y} = 0.
\]

So we need to solve

\[
\frac{dy}{dx} = \frac{3x^2 - 1}{2y}.
\]

This is separable:

\[
2y \, dy = 3x^2 - 1 \, dx.
\]
\[
\int 2y \, dy = \int 3x^2 - 1 \, dx
\]
\[
y^2 = x^3 - x + C.
\]

We can put this in the form \( y^2 - x^3 + x = C \) and hence
\[
u(x, y) = f \left( y^2 - x^3 + x \right).
\]

**Remark:** This technique can be generalized to PDEs of the form
\[
A(x, y) \frac{\partial u}{\partial x} + B(x, y) \frac{\partial u}{\partial y} = C(x, y, u).
\]
Example

**Solve** \[\frac{\partial u}{\partial x} + x \frac{\partial u}{\partial y} = u.\]

As above, along a curve \(y = y(x)\) we have

\[
\frac{d}{dx} u(x, y(x)) = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{dy}{dx}.
\]

Comparison with the original PDE gives the characteristic ODEs

\[
\frac{dy}{dx} = x,
\]

\[
\frac{d}{dx} u(x, y(x)) = u(x, y(x)).
\]
The first tells us that

\[ y = \frac{x^2}{2} + y(0), \]

and the second that

\[ u(x, y(x)) = u(0, y(0))e^x = f(y(0))e^x. \]

Combining these gives

\[ u(x, y) = f \left( y - \frac{x^2}{2} \right) e^x. \]

---

\(^1\)Recall that the solution to the ODE \( \frac{dw}{dx} = kw \) is \( w = Ce^{kx} \). Since \( w(0) = Ce^0 = C \), we can write this as \( w = w(0)e^{kx} \).
Consider a first order PDE of the form

$$A(x, y) \frac{\partial u}{\partial x} + B(x, y) \frac{\partial u}{\partial y} = C(x, y, u).$$ (5)

- When $A(x, y)$ and $B(x, y)$ are constants, a linear change of variables can be used to convert (5) into an “ODE.”

- In general, the method of characteristics yields a system of ODEs equivalent to (5).

In principle, these ODEs can always be solved completely to give the general solution to (5).