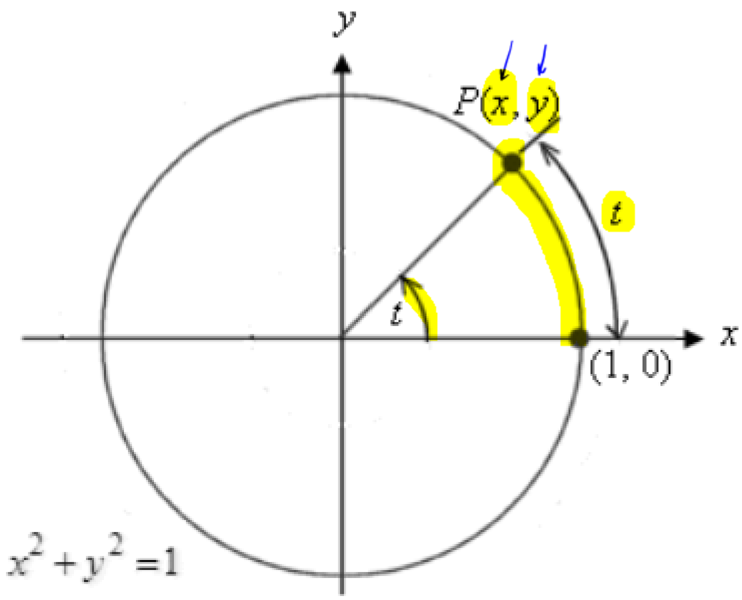
Section 5.1 Trigonometric Functions of Real Numbers

In calculus and in the sciences many of the applications of the trigonometric functions require that the inputs be real numbers, rather than angles. By making this small but crucial change in our viewpoint, we can define the trigonometric functions in such a way that the inputs are real numbers. The definitions of the trig functions, and the identities that we have already met (and will meet later) will remain the same, and will be valid whether the inputs are angles or real numbers.

Definition of the Trigonometric Functions of Real Numbers

Let $P(x, y)$ be a point on the unit circle $x^2 + y^2 = 1$ whose arc length from the point $(1,0)$ is t.



Then the six trigonometric functions of the real number t is defined as:

$\sin(t) = y$ $\qquad\qquad$ $\csc(t) = \dfrac{1}{y}$ $\quad (y \neq 0)$

$\cos(t) = x$ $\qquad\qquad$ $\sec(t) = \dfrac{1}{x}$ $\quad (x \neq 0)$

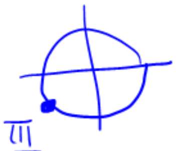
$\tan(t) = \dfrac{y}{x}$ $\qquad\qquad$ $\cot(t) = \dfrac{x}{y}$ $\quad (y \neq 0)$

Here are some identities you need to know:

Reciprocal Identities	**Pythagorean Identities**
$\tan(t)=\dfrac{\sin(t)}{\cos(t)}$ $\cot(t)=\dfrac{\cos(t)}{\sin(t)}$ $\sec(t)=\dfrac{1}{\cos(t)},\quad \cos(t)\neq 0$ $\csc(t)=\dfrac{1}{\sin(t)},\quad \sin(t)\neq 0$ $\cot(t)=\dfrac{1}{\tan(t)},\quad \tan(t)\neq 0$	$\sin^2(t)+\cos^2(t)=1$ $1+\tan^2(t)=\sec^2(t)$ $1+\cot^2(t)=\csc^2(t)$

sec: $-$
$\Uparrow$
cos: $-$
$\Uparrow$
tan t: $+$

sin t: $-$
$\Downarrow$



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Example 1: If $\tan(t) = \frac{1}{5}$ and $\boxed{\sin(t) < 0}$, find $\sec(t)$.

$$1 + \tan^2(t) = \sec^2(t) \Rightarrow 1 + \left(\frac{1}{5}\right)^2 = \sec^2(t)$$

$$\frac{1}{1} + \frac{1}{25} = \frac{26}{25} \Rightarrow \sec^2(t) = \frac{26}{25}$$

$$\Rightarrow \sec(t) = \pm\sqrt{\frac{26}{25}}$$

$$\Rightarrow \boxed{\sec(t) = -\frac{\sqrt{26}}{5}}$$

Example 2: Given that $\underbrace{\cos(t) = -\frac{1}{4}}_{\cos:\,-}$ and $\underbrace{\tan(t) > 0}_{\substack{\tan:\,+ \\ \sin:\,-}}$; find $\sin(t)$ and $\tan(t)$.

$$\sin^2 t + \cos^2 t = 1$$

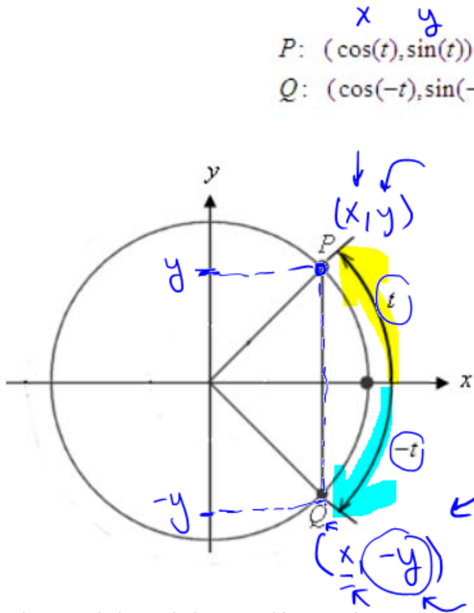
$$\sin^2 t = 1 - \cos^2 t = 1 - \left(-\frac{1}{4}\right)^2 = \frac{1}{1} - \frac{1}{16} = \frac{15}{16}$$

$$\Rightarrow \sin(t) = \pm\sqrt{\frac{15}{16}} = -\frac{\sqrt{15}}{4}$$

$$\tan(t) = \frac{\sin t}{\cos t} = \frac{-\frac{\sqrt{15}}{4}}{-\frac{1}{4}} = \frac{\sqrt{15}}{\cancel{4}} \cdot \frac{\cancel{4}}{1} = \boxed{\sqrt{15}}$$

Opposite Angle Identities

The positive direction from the point (1,0) is the counterclockwise direction and the negative direction is the clockwise direction.



$$\sin(-t) = -\sin(t)$$
$$\cos(-t) = \cos(t)$$
$$\tan(-t) = -\tan(t)$$
$$\cot(-t) = -\cot(t)$$
$$\sec(-t) = \sec(t)$$
$$\csc(-t) = -\csc(t)$$

$$\tan(-t) = \frac{\sin(-t)}{\cos(-t)} = \frac{-\sin t}{\cos t} = -\tan t$$

$f(-x) = -f(x)$

These identities tell us that:
Sine, tangent, cotangent and cosecant are ODD functions;
Cosine and Secant are EVEN functions.

$f(-x) = f(x)$

These identities hold true both for angles and real numbers.

$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

Example 3: Use the opposite-angle identities to find:

a. $\sin\left(-\frac{2\pi}{3}\right) = -\underbrace{\sin\left(\frac{2\pi}{3}\right)} = \boxed{-\frac{\sqrt{3}}{2}}$

b. $\cos\left(-\frac{5\pi}{6}\right) = \cos\left(\frac{5\pi}{6}\right) = \boxed{-\frac{\sqrt{3}}{2}}$

unit circle

c. $\tan\left(-\frac{\pi}{4}\right) + \cot\left(-\frac{3\pi}{4}\right) = -1 + 1 = \boxed{0}$

$\tan\left(-\frac{\pi}{4}\right) = -\underbrace{\tan\left(\frac{\pi}{4}\right)} = -1$

$\cot\left(-\frac{3\pi}{4}\right) = -\underbrace{\cot\left(\frac{3\pi}{4}\right)}_{-1} = -(-1) = 1$

add: $0 < x < \frac{\pi}{2}$ (quadrant I)

Example 4: Given: $\sin(x) = \frac{12}{13}$, find the value of: $\sin(-x) + \cos(-x) = -\sin x + \cos x$

$= -\frac{12}{13} + \frac{5}{13}$

$= \boxed{\frac{-7}{13}}$

$\sin(-x) = -\sin(x)$

$\cos(-x) = \cos x$

$\cos^2 x = 1 - \sin^2 x = 1 - \left(\frac{12}{13}\right)^2$

$= 1 - \frac{144}{169}$

$= \frac{169-144}{169}$

$= \frac{25}{169}$

$\cos x = \pm\sqrt{\frac{25}{169}} = + \frac{5}{13}$

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Question#1: Which of the following statements is/are true?

I. $1 + \tan^2(t) = \sec^2(t)$

II. $\cos(-x) = -\cos(x)$

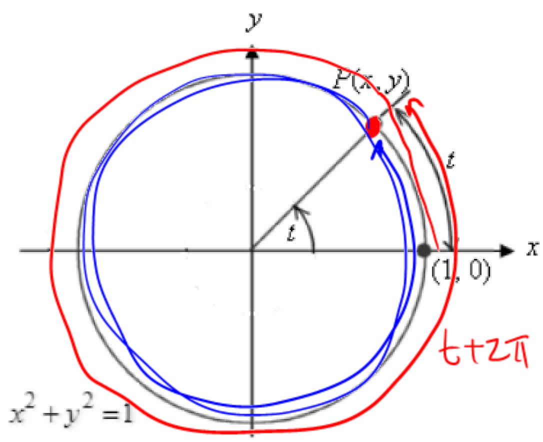
III. $\sin(x)\csc(x) = 1$

Periodic functions:
A function is periodic if there is some number *p* such that $f(x+p)=f(x)$. The smallest such number is called the "period" of this function.

Periodicity of Sine and Cosine Functions

$2\pi \cdot r = 2\pi \cdot 1 = 2\pi$

The circumference of the unit circle is 2π. Thus, if we start at a point P and travel for 2π units, we come back to that point P.
The period for sine and cosine functions is 2π.



$\sin(t+2\pi)=\sin(t)$
$\cos(t+2\pi)=\cos(t)$

$\sin(t+4\pi)=\sin(t)$
$\cos(t+4\pi)=\cos(t)$

$\sin(t+20\pi)=\sin(t)$

More generally:

even

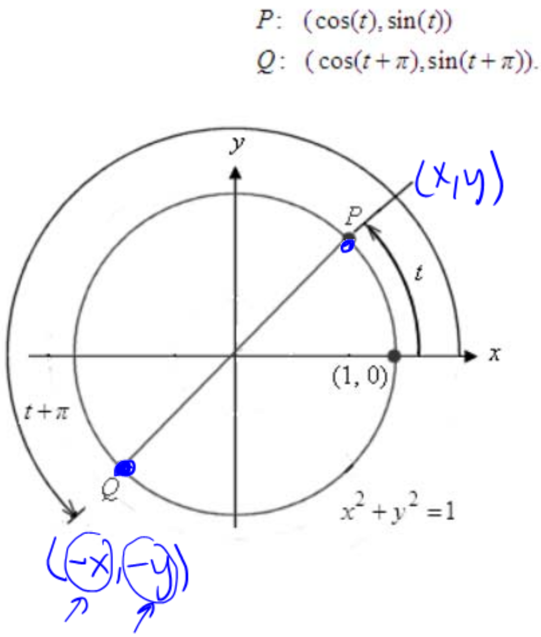
$\sin(t+2k\pi)=\sin(t)$
$\cos(t+2k\pi)=\cos(t)$

for all real numbers *t* and all integers *k*.

Periodicity of Tangent and Cotangent Functions

The tangent and cotangent functions are also periodic functions. However, these functions repeat themselves after each π .

$\frac{-}{-} = +$



$$\tan(t+\pi) = \frac{\sin(t+\pi)}{\cos(t+\pi)} = \frac{-\sin(t)}{-\cos(t)} = \tan(t)$$

$$\tan(t+\pi) = \frac{-y}{-x} = \frac{y}{x} = \tan(t)$$

$$\tan(t+5\pi) = \tan(t)$$

$$\cot(t+11\pi) = \cot(t)$$

$$\cot(t+12\pi) = \cot(t)$$

So

$$\tan(t+\pi) = \tan(t)$$
$$\cot(t+\pi) = \cot(t)$$

more generally

$$\boxed{\begin{array}{c}\tan(t+k\pi) = \tan(t) \\ \cot(t+k\pi) = \cot(t)\end{array}}$$

for all real numbers t and all integers k.

Remark: The period for the sine and cosine functions is 2π while the period for the tangent and cotangent functions is π.

Since secant and cosecant functions are the reciprocal functions, they will follow the same periodicity rules as sine and cosine.

$$\sec(t + 2\pi k) = \sec(t)$$
$$\csc(t + 2\pi k) = \csc(t)$$

for all real numbers t and all integers k.

We will use the identities and periodicity to evaluate trig functions of real numbers.

Periodicity

$$\begin{aligned} &\sin(t+2k\pi)=\sin(t) && \tan(t+k\pi)=\tan(t) \\ &\cos(t+2k\pi)=\cos(t) && \cot(t+k\pi)=\cot(t) \\ &\sec(t+2\pi k)=\sec(t) \\ &\csc(t+2\pi k)=\csc(t) \end{aligned}$$

(for all real numbers t and all integers k.)

Example 4: Evaluate $\tan\left(\frac{15\pi}{4}\right) = \tan\left(3\pi + \frac{3\pi}{4}\right) = \tan\left(\frac{3\pi}{4}\right) = \boxed{-1}$

$\frac{15}{4} = 3\frac{3}{4}$

even!

Example 5: Evaluate $\cos\left(\frac{25\pi}{6}\right) = \cos\left(4\pi + \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = \boxed{\frac{\sqrt{3}}{2}}$

$\frac{25}{6} = 4\frac{1}{6}$

$4 + \frac{1}{6}$

big!

even!

Example 6: Evaluate $\sin\left(\frac{19\pi}{3}\right) = \sin\left(6\pi + \frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \boxed{\frac{\sqrt{3}}{2}}$

$19\frac{\pi}{3} = \frac{6\pi}{1} + \frac{\pi}{3}$

Example 7: Evaluate $\sin\left(\frac{-20\pi}{3}\right) = -\sin\left(\frac{20\pi}{3}\right) = \boxed{-\frac{\sqrt{3}}{2}}$

$\sin\left(\frac{20\pi}{3}\right) = \sin\left(6\pi + \frac{2\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

even!

$6\pi + \frac{2\pi}{3}$

$\frac{20}{3} = 6\frac{2}{3}$

Example: Evaluate $\cos(21\pi)+\sin(40\pi)+\tan(5\pi) = -1+0+0 = \boxed{-1}$

$\cos(21\pi) = \cos(\pi + 20\pi) = \cos(\pi) = -1$

$\sin(40\pi) = \sin(0 + 40\pi) = \sin(0) = 0$

$\tan(5\pi) = \tan(0+5\pi) = \tan(0) = 0$

Remark: If the real number is a multiple of π :

$\cos(80\pi) = 1$

even

* $\sin(n\pi)=0$ **if n is an integer.**

$\cos(n\pi)=1$ **if n is an EVEN integer.**

$\cos(n\pi)=-1$ **if n is an ODD integer.**

$\cos(57\pi) = -1$

odd

Reason:

$\cos(2k\pi)=\cos(0+2k\pi)=\cos(0)=1$

$\cos\big((2k+1)\pi\big)=\cos(\pi+2k\pi)=\cos(\pi)=-1$

$\sin(85\pi) = 0$

Example 8: Evaluate $\dfrac{\cos\left(\frac{19\pi}{3}\right)\tan\left(\frac{21\pi}{4}\right)}{\cos(8\pi)\sin\left(\frac{25\pi}{6}\right)} = \dfrac{\frac{1}{2}\cdot 1}{1\cdot\frac{1}{2}} = \boxed{1}$

$\cos\left(\frac{19\pi}{3}\right) = \cos\left(6\pi + \frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$\tan\left(\frac{21\pi}{4}\right) = \tan\left(5\pi + \frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) = 1$

$\cos(8\pi) = 1 \qquad \cos(0 + 8\pi)$

$\sin\left(\frac{25\pi}{6}\right) = \sin\left(4\pi + \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

Example 9: Evaluate $\cot\left(\frac{21\pi}{4}\right)+\frac{\tan\left(\frac{17\pi}{4}\right)}{\cos(11\pi)\sin\left(\frac{17\pi}{2}\right)}$ $= 1 + \frac{1}{(-1)\cdot 1}$

$= 1 - 1 = \boxed{0}$

$\cot\left(\frac{21\pi}{4}\right) = \cot\left(5\pi + \frac{\pi}{4}\right) = \cot\left(\frac{\pi}{4}\right) = 1$

$\tan\left(\frac{17\pi}{4}\right) = \tan\left(4\pi + \frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) = 1$

$\cos(\underline{\underline{11}}\pi) = -1$

odd

$\sin\left(17\frac{\pi}{2}\right) = \sin\left(8\pi + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$

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Question#2: **Evaluate:** $\cos(19\pi)+\sin\left(-\frac{\pi}{6}\right)+\tan\left(\frac{41\pi}{4}\right)$

Simplifying Trigonometric Expressions Using Identities

Example 1: Simplify: $\cot(-t)\sec(-t)$

$= -\cot(t)\cdot\sec(t)$

$= -\dfrac{\cos(t)}{\sin(t)}\cdot\dfrac{1}{\cos(t)}$

$= -\dfrac{1}{\sin(t)}$

$= \boxed{-\csc(t)}$

Example 2: Simplify: $\cos(-t)+\cos(-t)\tan^2(-t)$

$\cos(-t) = \cos(t)$

$\tan(-t) = -\tan(t)$

$\tan^2(-t) = [-\tan(t)]^2$

$= \tan^2(t)$

$= \cos(t) + \cos(t)\cdot\tan^2(t)$

$= \cos(t)\cdot[1+\tan^2(t)]$

$= \cos(t)\cdot\sec^2(t)$

$= \cos(t)\cdot\dfrac{1}{\cos^2(t)}$

$= \dfrac{1}{\cos(t)} = \boxed{\sec(t)}$

Example 3: Simplify: $\dfrac{\sin(\theta+6\pi)\csc(\theta-2\pi)}{1+\tan^2(\theta+5\pi)}$

$\tan(\theta+5\pi) = \tan(\theta)$

$$= \frac{\sin(\theta)\cdot\csc(\theta)}{1+\tan^2(\theta)}$$

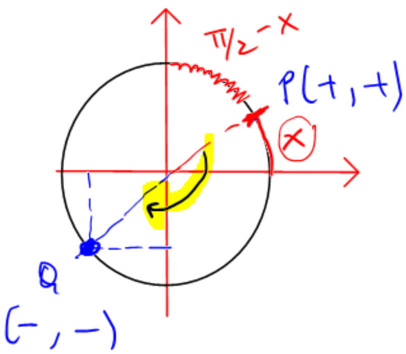
$\sin\theta\cdot\csc\theta = 1$

$$= \frac{1}{\sec^2(\theta)} = \frac{1}{\frac{1}{\cos^2(\theta)}} = \boxed{\cos^2\theta}$$

Quad I

Example 4: Given $\cos(x) = \frac{5}{13}$ and $0 < x < \frac{\pi}{2}$, evaluate the following

expression: $\underbrace{\sin\left(\frac{\pi}{2} - x\right)}_{\cos(x)} + \underbrace{\sin(x - \pi)}_{-\sin(x)}$



$\cos(x) + -\sin(x)$

$= \frac{5}{13} - \frac{12}{13}$

$= \boxed{\frac{-7}{13}}$

$\sin^2 x = 1 - \cos^2 x$

$= 1 - \frac{25}{169}$

$= \frac{144}{169}$

$\sin x = \pm\sqrt{\frac{144}{169}}$

$\sin x = \frac{12}{13}$

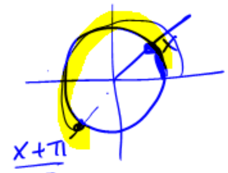
x & $\frac{\pi}{2} - x$

Remark:
Recall the fact about "complementary angles":

$$\begin{cases} \sin\left(\frac{\pi}{2} - x\right) = \cos(x) \\ \cos\left(\frac{\pi}{2} - x\right) = \sin(x) \\ \tan\left(\frac{\pi}{2} - x\right) = \cot(x) \end{cases}$$

Example 5: Given $\boxed{\sin(x) = \frac{4}{5}}$ and $0 < x < \frac{\pi}{2}$, evaluate the following expression: $\sin(x+11\pi)+\cos(x-9\pi)+\tan(x+17\pi)$

quad: I



$\sin(x+11\pi) = \sin(x+\pi+10\pi) = \sin(x+\pi) = -\sin(x)$

$\cos(x-9\pi) = \cos(x+\pi-10\pi) = \cos(x+\pi) = -\cos(x)$

$\tan(x+17\pi) = \tan(x)$

$-\sin x + -\cos x + \tan x$

$\tan x = \frac{\sin x}{\cos x} = \frac{4/5}{3/5} = \frac{4}{3}$

$= -\frac{4}{5} + -\frac{3}{5} + \frac{4}{3}$

$= -\frac{7}{5} + \frac{4}{3} = \frac{-21+20}{15} = \boxed{-\frac{1}{15}}$

(3) (5)