Section 7.3 - The Law of Sines and the Law of Cosines

Sometimes you will need to solve a triangle that is not a right triangle. This type of triangle is called an oblique triangle. To solve an oblique triangle you will not be able to use right triangle trigonometry. Instead, you will use the Law of Sines and/or the Law of Cosines.

You will typically be given three parts of the triangle and you will be asked to find the other three. The approach you will take to the problem will depend on the information that is given.

If you are given SSS (the lengths of all three sides) or SAS (the lengths of two sides and the measure of the included angle), you will use the Law of Cosines to solve the triangle.

If you are given SAA (the measures of two angles and one side) or SSA (the measures of two sides and the measure of an angle that is not the included angle), you will use the Law of Sines to solve the triangle.

Recall from your geometry course that SSA does not necessarily determine a triangle. We will need to take special care when this is the given information.

Please read this! Geometry

Here are some facts about solving triangles that may be helpful in this section:

If you are given SSS, SAS or SAA, the information determines a unique triangle.

If you are given SSA, the information given may determine 0, 1 or 2 triangles. This is called the "ambiguous case" of the law of sines. If this is the information you are given, you will have some additional work to do.

Since you will have three pieces of information to find when solving a triangle, it is possible for you to use both the Law of Sines and the Law of Cosines in the same problem.

When drawing a triangle, the measure of the largest angle is opposite the longest side; the measure of the middle-sized angle is opposite the middle-sized side; and the measure of the smallest angle is opposite the shortest side.

Suppose a, b and c are suggested to be the lengths of the three sides of a triangle. Suppose that c is the biggest of the three measures. In order for a, b and c to form a triangle, this inequality must be true: $a + b > c$. So, the sum of the two smaller sides must be greater than the third side.

An obtuse triangle is a triangle which has one angle that is greater than 90°. An acute triangle is a triangle in which all three angles measure less than 90°.

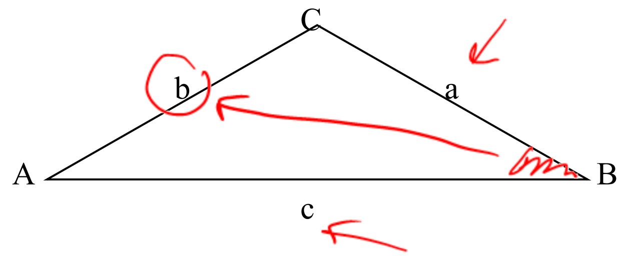
If you are given the lengths of the three sides of a triangle, where $c > a$ and $c > b$, you can determine if the triangle is obtuse or acute using the following:

If $a^2 + b^2 > c^2$, the triangle is an acute triangle.
If $a^2 + b^2 < c^2$, the triangle is an obtuse triangle.

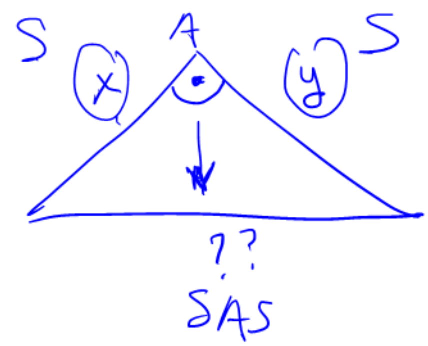
Your first task will be to analyze the given information to determine which formula to use. You should sketch the triangle and label it with the given information to help you see what you need to find. If you have a choice, it is usually best to find the largest angle first.

Here's the **Law of Cosines.** In any triangle *ABC*,

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

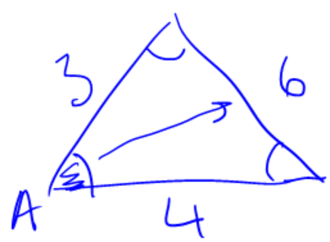


This law is used for SAS or SSS cases!

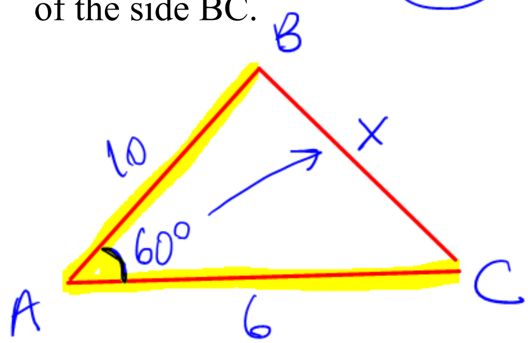


SAS: Two sides and the included angle are given
(use LOC to find the third side)

SSS: Three sides are given
(use LOC to find the angles)



Example 1: In ΔABC, AC = 6, AB = 10, and $m\angle A = 60^0$. Find the length of the side BC.



Given: SAS

use: LOC

$$x^2 = 6^2 + 10^2 - 2 \cdot 6 \cdot 10 \cdot \cos(60^0)$$

$$x^2 = \underbrace{36 + 100} - \cancel{2} \cdot 6 \cdot 10 \cdot \frac{1}{\cancel{2}}$$

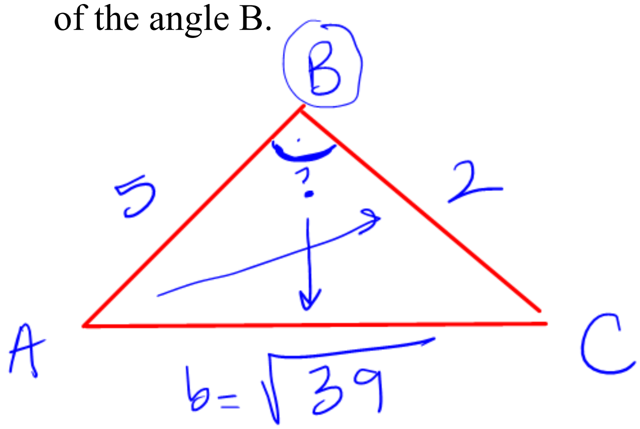
$$x^2 = 136 - 60$$

$$x^2 = 76 \quad \Rightarrow \quad x = \sqrt{76}$$

$$\boxed{x = 2 \cdot \sqrt{19}}$$

Example 2: In ΔABC, BC = 2, AB = 5, and AC=$\sqrt{39}$. Find the measure of the angle B.

Given: SSS



$$b^2 = a^2 + c^2 - 2ac\cdot\cos(B)$$

$$(\sqrt{39})^2 = 2^2 + 5^2 - 2\cdot 2\cdot 5\cdot\cos(B)$$

$$39 = 4 + 25 - 20\cdot\cos(B)$$

$$39 = 29 - 20\cdot\cos(B)$$

$$-29 \qquad -29$$

$$10 = -20\cos(B) \Rightarrow \cos(B) = \frac{10}{-20} = -\frac{1}{2}$$

$$\Rightarrow \cos(B) = -\frac{1}{2} \xrightarrow[\text{unit circle}]{} \boxed{m\angle B = 120^\circ}$$

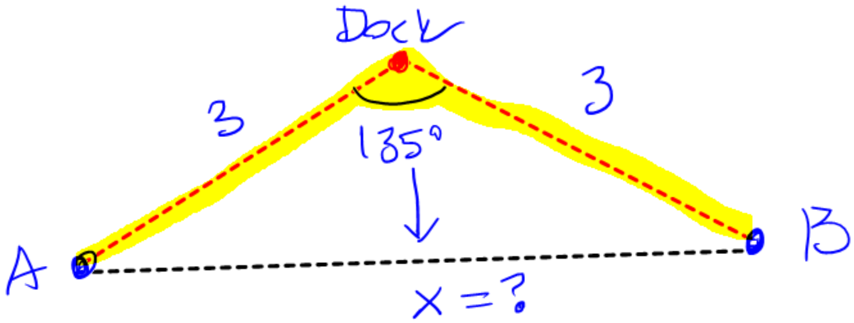
What if

$$\cos(B) = \frac{1}{5} \Rightarrow B = \arccos\left(\frac{1}{5}\right) = \ldots$$

calculator

not on unit circle

Example 3: Two sailboats leave the same dock together traveling on courses that have an angle of 135° between them. If each sailboat has traveled 3 miles, how far apart are the sailboats from each other?



Given: SAS

Use: LoC

$$x^2 = 3^2 + 3^2 - 2 \cdot 3 \cdot 3 \cdot \cos(135°)$$

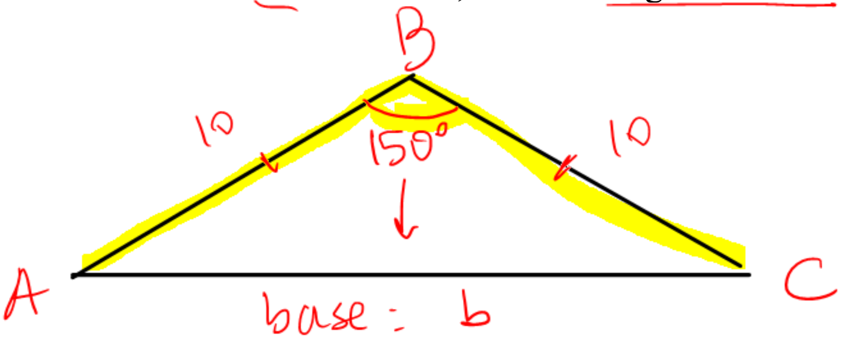
$$x^2 = 9 + 9 - \cancel{2} \cdot 3 \cdot 3 \cdot \frac{-\sqrt{2}}{\cancel{2}}$$

$$x^2 = \underbrace{9 + 9} + 9\sqrt{2} = 18 + 9\sqrt{2}$$

$$x^2 = 18 + 9\sqrt{2}$$

$$\Rightarrow \quad x = \sqrt{18 + 9\sqrt{2}}$$

Example 4: An isosceles triangle has a vertex angle of 150^0. If the equal sides are 10 units each, find the length of the base.



Given: SAS

Use: LOC

$$b^2 = 10^2 + 10^2 - 2\cdot \underbrace{10\cdot 10}\cdot \underbrace{\cos(150^\circ)}$$

$$b^2 = 100 + 100 - \cancel{2}\cdot 100 \cdot \frac{-\sqrt{3}}{\cancel{2}}$$

$$b^2 = 200 + 100\sqrt{3}$$

$$\Rightarrow \boxed{b = \sqrt{200 + 100\sqrt{3}}}$$

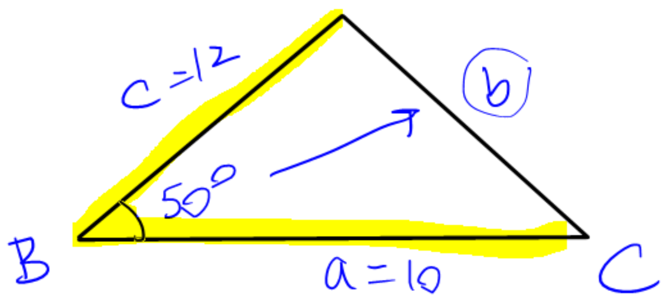
pull out 100

$\sqrt{100} = 10$

$$b = 10\cdot\sqrt{2+\sqrt{3}}$$

Some problem might require to use a calculator. Your exam questions will not require the use of a calculator.

Example 5: In ΔABC, $\angle B = 50°$, $a = 10$ and $c = 12$. Find the length of AC. Round your answer to the nearest tenth.



Given: SAS

Use: LOC

$$b^2 = a^2 + c^2 - 2ac \cdot \cos(B)$$

$$b^2 = 10^2 + 12^2 - 2 \cdot 10 \cdot 12 \cdot \cos(50°)$$

Not on unit circle

calculator

$$b^2 = 100 + 144 - 240 \cdot \cos(50°)$$

$$b^2 = 244 - 240 \cdot \cos(50°)$$

$$b^2 = 89.73 \Rightarrow b = \sqrt{89.73}$$

$$b = 9.47 = \boxed{9.5}$$

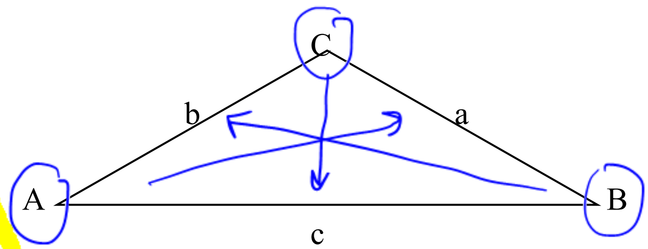
POPPER for Section 7.3

Question#1: In triangle ABC, AB=6 and AC=4, and the measure of angle A is 60^0 . Find the length of BC.

THE LAW OF SINES

Here's the **Law of Sines**. In any triangle *ABC*,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$



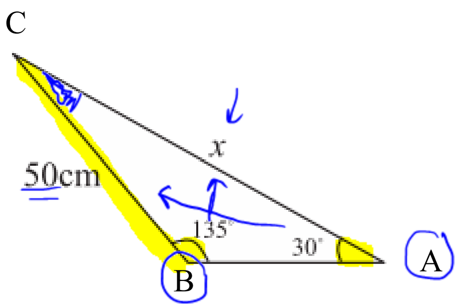
This law is USED FOR SAA, SSA cases!

SAA: One side and two angles are given

SSA: Two sides and an angle opposite to one of those sides are given

Example 6: Find x.

Given: SAA
use: LOS



$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b}$$

$$\frac{\sin(30^\circ)}{50} = \frac{\sin(135^\circ)}{x}$$

$$x \cdot \sin(30^\circ) = 50 \cdot \sin(135^\circ)$$

$$\Rightarrow \quad x \cdot \frac{1}{2} = 50 \cdot \frac{\sqrt{2}}{\cancel{2}}$$

Multiply by 2:

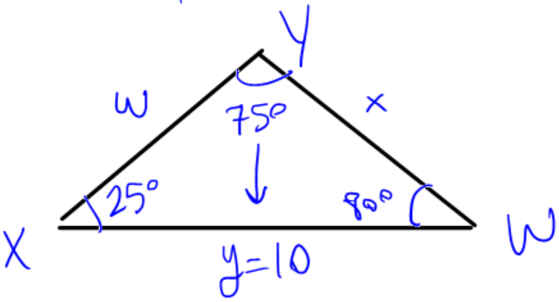
$$\boxed{x = 50 \cdot \sqrt{2}}$$

$25 + 75 = 100$

$180 - 100 = 80^\circ$

$\angle W$

Example 7: In ΔXYW, $m\angle X = 25^\circ$, $m\angle Y = 75^\circ$ and $y = 10$. Solve the triangle. (Use a calculator and round to the nearest tenth.)



LOS

$$\frac{\sin X}{x} = \frac{\sin Y}{y} = \frac{\sin W}{w}$$

$$\frac{\sin(25^\circ)}{x} = \frac{\sin(75^\circ)}{10} = \frac{\sin(80^\circ)}{w}$$

cross product:

$$x \cdot \sin(75^\circ) = 10 \cdot \sin(25^\circ)$$

$$\Rightarrow \quad x = \frac{10 \cdot \sin(25^\circ)}{\sin(75^\circ)} = 4.38$$

$$x = \boxed{4.4}$$

$$\frac{\sin(75^\circ)}{10} \times \frac{\sin(80^\circ)}{w}$$

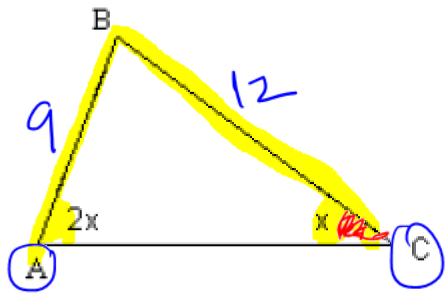
$$w \cdot \sin(75^\circ) = 10 \cdot \sin(80^\circ)$$

calc.

$$w = \frac{10 \cdot \sin(80^\circ)}{\sin(75^\circ)} = 10.2$$

$$\boxed{w = 10.2}$$

Example 8: In the triangle below, AB=9, BC=12. Find the value of x. (You may use inverse trig notation to express your answer.)



LOS

$$\frac{\sin(A)}{a} = \frac{\sin(C)}{c}$$

$$\frac{\sin(2x)}{12} = \frac{\sin(x)}{9}$$

$$\Rightarrow \quad 9 \cdot \sin(2x) = 12 \cdot \sin(x)$$

formula (double angle)

$$\cancel{9}^{3} \cdot \cancel{2} \cdot \cancel{\sin(x)} \cdot \cos(x) = \cancel{\cancel{12}}^{2} \cdot \cancel{\sin(x)}$$

(we are sure that $\sin x \neq 0$)

$$\Rightarrow \quad 3 \cdot \cos(x) = 2$$

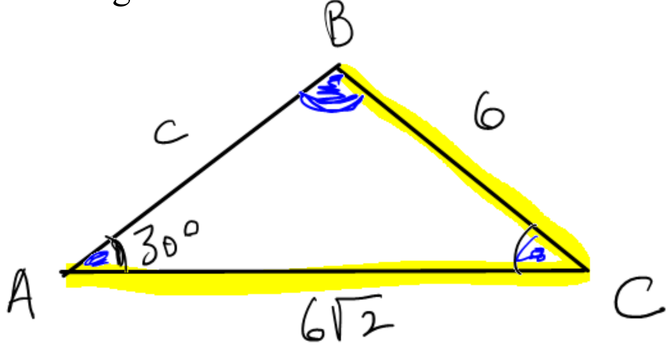
$$\Rightarrow \quad \cos(x) = \frac{2}{3} \Rightarrow \boxed{x = \arccos\left(\frac{2}{3}\right)}$$

calc: $x = 48.19^0$

POPPER for Section 7.3

Question#2: In triangle ABC, $m\angle A = 30^0, m\angle B = 45^0, a = 12$ **. Find the length of AC=b.**

Example 9: In ΔABC, $m\angle A = 30°$, $b = 6\sqrt{2}$ and $a = 6$. Find the measures of angles B and C.



given: SSA (⚠!)

use: LOS

$$\boxed{\frac{\sin(30°)}{6} = \frac{\sin(B)}{6\sqrt{2}}} = \frac{\sin(C)}{c}$$

cross product: $6\sqrt{2} \cdot \sin(30°) = 6 \cdot \sin(B)$

$$\Rightarrow \sin(B) = \frac{\cancel{6}\sqrt{2} \cdot \sin(30°)}{\cancel{6}} = \sqrt{2} \cdot \frac{1}{2} = \frac{\sqrt{2}}{2}$$

$$\Rightarrow \sin(B) = \frac{\sqrt{2}}{2} \Rightarrow \begin{cases} m\angle B = \boxed{45°} \\ \text{OR} \\ m\angle B = 135° \end{cases}$$

$m\angle A + m\angle B + m\angle C = 180°$

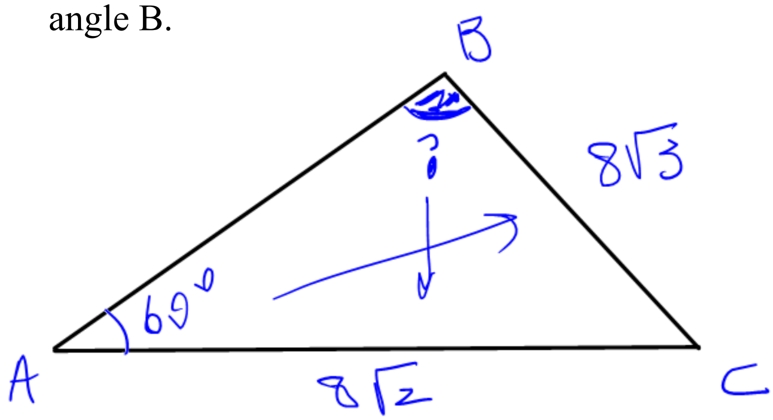
Case 1: $m\angle B = 45° \Rightarrow 30° + 45° + m\angle C = 180°$

$m\angle C = 180 - 75 = \boxed{105°}$

Case 2: $m\angle B = 135° \Rightarrow 30° + 135° + m\angle C = 180$

$m\angle C = 15°$

Example 10: In triangle ABC, the measure of angle A is 60^0, the length of BC is $8\sqrt{3}$ and the length of AC is $8\sqrt{2}$. Find all possible measures for angle B.



Given: SSA

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b}$$

$$\frac{\sin(60^0)}{8\sqrt{3}} = \frac{\sin(B)}{8\sqrt{2}}$$

$$\Rightarrow \cancel{8}\sqrt{2}\cdot\sin(60^0) = \cancel{8}\sqrt{3}\cdot\sin(B)$$

$$\Rightarrow \sin(B) = \frac{\sqrt{2}\cdot\sin(60^0)}{\sqrt{3}} = \frac{\sqrt{2}\cdot\cancel{\sqrt{3}}/2}{\cancel{\sqrt{3}}}$$

$$\Rightarrow \sin(B) = \frac{\sqrt{2}}{2}$$

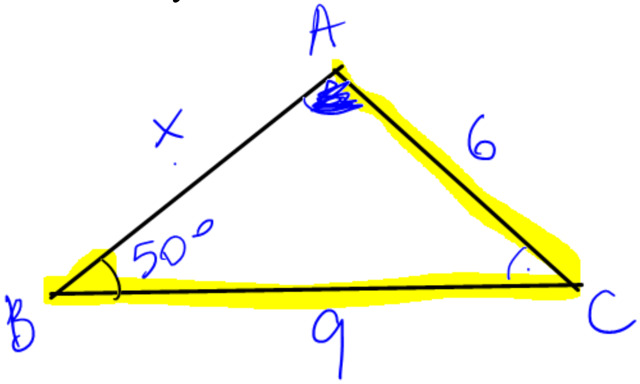
$$\Rightarrow \boxed{B = 45^0} \quad \text{or} \quad \cancel{B = 135^0}$$

Only 45degrees would be acceptable because the sum of all angles in a triangle should be 180degrees. If we go with B=135, then 60+135>180.

Example 11: In ΔABC, $\angle B = 50°$, $b = 6$ and $a = 9$. Find the length of AB. Round your answer to the nearest tenth.

given: SSA

use: LOS



$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b}$$

$$\frac{\sin(A)}{9} = \frac{\sin(50°)}{6}$$

$$\Rightarrow \sin(A) = \frac{9 \cdot \sin(50°)}{6} = 1.15$$

$$\Rightarrow \sin(A) = 1.15$$

Think!

$1.15 > 1$

$-1 \leq \sin A \leq 1$

This is NOT <u>possible</u>.

No such triangle exists.

Note: *SSA* case is called the **ambiguous case** of the law of sines. There may be two solutions, one solution, or no solutions. You should throw out the results that don't make sense. That is, if $\sin A > 1$ or the angles add up to more than 180^0.

SSA Case (Two sides and an angle opposite to those sides)

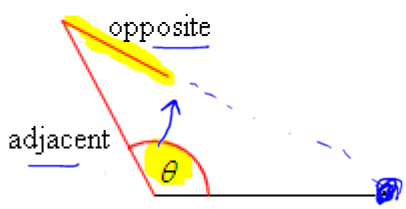
In the last case (SSA) for solving oblique triangles, two sides and the angle opposite one of those sides are given. Suppose that θ is the given angle. The other given side must be adjacent to θ. We consider several cases.

Case 1: Suppose that $\theta > 90^\circ$. Two possibilites arise.

(a) If opposite $\leq$ adjacent, no triangle is formed. (There is no solution.)

(b) If opposite > adjacent, one triangle is formed. (Use the Law of Sines.)

Case 1 Part (a)

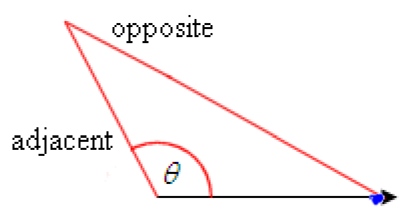


$\theta > 90^\circ$

opposite $\leq$ adjacent

No

Case 1 Part (b)



$\theta > 90^\circ$

opposite > adjacent

1 triangle

Case 2: Suppose that $\theta < 90^\circ$. Four possibilites arise. Let h be the length of the altitude of the triangle drawn from the vertex that connects the opposite and adjacent sides.

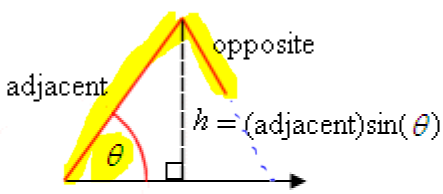
(a) If opposite $< h <$ adjacent, no triangle is formed. (There is no solution.)

(b) If opposite $= h <$ adjacent, one right triangle is formed. (Use right triangle trigonometry to solve the triangle as in Section 7.1.)

(c) If $< h <$ opposite $<$ adjacent, two different triangles are formed. This is called the ambiguous case. (Use the Law of Sines to find two solutions.)

(d) If opposite $\geq$ adjacent, one triangle is formed. (Use the Law of Sines.)

Case 2 Part (a)

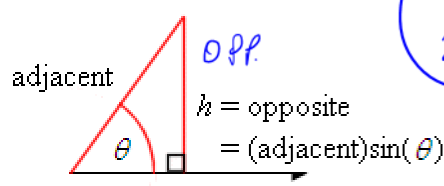


$\theta < 90^\circ$

opposite $< h <$ adjacent

No such triangle

Case 2 Part (b)

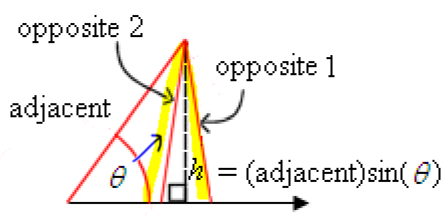


1

$\theta < 90^\circ$

opposite $= h <$ adjacent

Case 2 Part (c)

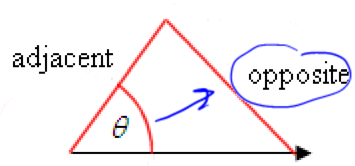


$\theta < 90^\circ$

$h <$ opposite $<$ adjacent

2

Case 2 Part (d)



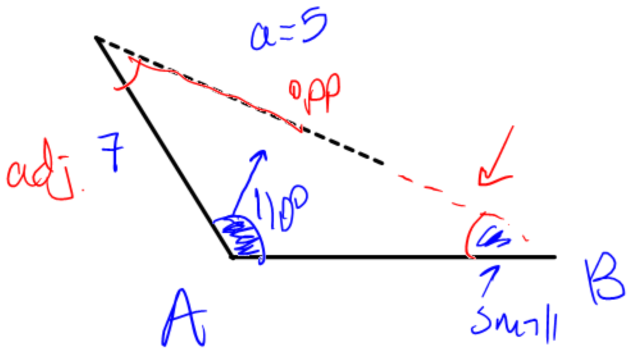
$\theta < 90^\circ$

opposite $\geq$ adjacent

1

Example 12: In $\Delta ABC, m\angle A = 110^0, a = 5, b = 7$. How many possible triangles are there?

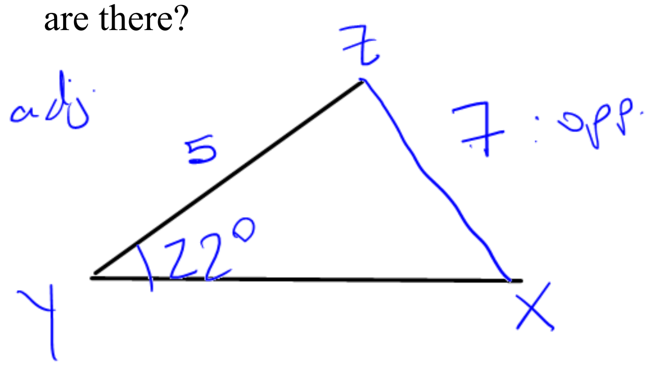
Opposite 110° should be the longest side



opposite < adjacent

No such triangle exists

Example 13: In ΔXYZ, $\angle Y = 22°$, $y = 7$, $x = 5$. How many possible triangles are there?

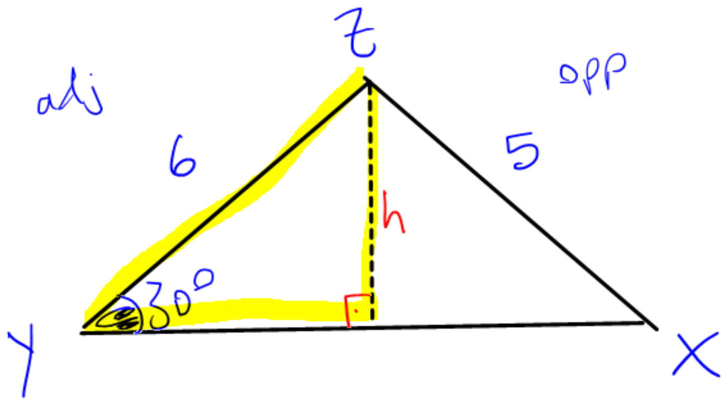


22° : acute
(case-2)

opposite > adjacent

<u>1 such triangle.</u>

Example 14: In $\Delta XYZ, m\angle \overset{Y}{\cancel{Z}} = 30^0, y = 5, x = 6$. How many possible triangles are there? If possible, find the other angles.



Case-2
$\theta: 30^\circ$ acute

opposite < adjacent

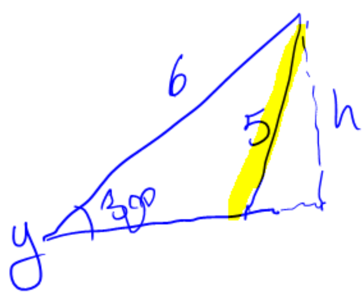
$$\sin(30^\circ) = \frac{h}{6}$$

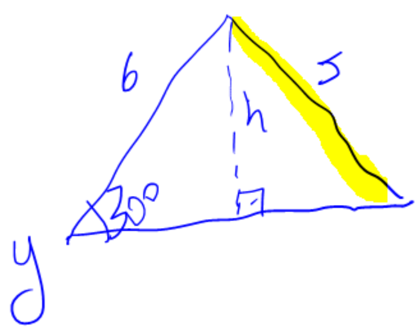
$$\frac{1}{2} = \frac{h}{6} \Rightarrow \boxed{h = 3}$$

adjacent: 6
opposite: 5
height: 3

height < opp. < adjacent

$\Rightarrow$ 2 such triangles.





You can use LOS to find other angles.

POPPER for Section 7.3

Question#3: In triangle ABC, the measure of angle A is 120^0, the length of BC is 9 and the length of AB is 4. How many such triangles are possible?