Math 1330 – Chapter 8 Conic Sections

In this chapter, we will study conic sections (or conics). It is helpful to know exactly what a conic section is. This topic is covered in Chapter 8 of the online text.

We start by looking at a double cone. Think of this as two “pointy” ice cream cones that are connected at the small tips:

To form a conic section, we’ll take this double cone and slice it with a plane. When we do this, we’ll get one of several different results.
1. Parabola
2. Ellipse
3. Circle
4. Hyperbola
5. Degenerate conic sections

You will find a nice tool on this website:

You can change the settings to see how these conic sections show up under certain conditions.
Next, we’ll look at parabolas. We previously studied parabolas as the graphs of quadratic functions. Now we will look at them as conic sections. There are a few differences. For example, when we studied quadratic functions, we saw that the graphs of the functions could open up or down. As we look at conic sections, we’ll see that the graphs of these second degree equations can also open left or right. So, not every parabola we’ll look at in this section will be a function.

We already know that the graph of a quadratic function \( f(x) = ax^2 + bx + c \) is a parabola.

But there is more to be learned about parabolas.
**Definition:** A *parabola* is the set of all points equally distant from a fixed line and a fixed point not on the line. The fixed line is called the *directrix*. The fixed point is called the *focus*.

The *axis*, or *axis of symmetry*, runs through the focus and is perpendicular to the directrix.

The *vertex* is the point *halfway between* the focus and the directrix.

We won’t be working with slanted parabolas, just with “horizontal” and “vertical” parabolas.

Visit this link for an interactive tool:
Basic “Vertical” Parabola:

Equation: $x^2 = 4py$

Focus: $(0, p)$

Directrix: $y = -p$

Focal Width: $4p$

Note: If $p$ is positive, it opens up
If $p$ is negative, it opens down
Basic "Horizontal" Parabola:

Equation: \( y^2 = 4px \)

Focus: \((p, 0)\)

Directrix: \(x = -p\)

Focal Width: \(4p\)

Note: If \(p\) is positive, it opens to the right
If \(p\) is negative, it opens left.

(In this case, the equation does not determine a function!)
POPPER for Section 8.1:

Question#1: What is the orientation of the parabola $x^2 = -10y$?
Graphing parabolas with **vertex at the origin**: 

- When you have an equation, look for \( x^2 \) or \( y^2 \).

- If it has \( x^2 \), it’s a “vertical” parabola. If it has \( y^2 \), it’s a “horizontal” parabola.

- Rearrange to look like \( y^2 = 4px \) or \( x^2 = 4py \). In other words, isolate the squared variable.

- Determine \( p \).

- Determine the direction it opens.
  - If \( p \) is positive, it opens right or up.
  - If \( p \) is negative, it opens left or down.

- Starting at the origin, place the focus \( p \) units to the inside of the parabola. Place the directrix \( p \) units to the outside of the parabola.

- Use the focal width \( |4p| \) (2 \( p \) on each side) to make the parabola the correct width at the focus.
Example 1: Write $y^2 - 20x = 0$ in standard form and graph it.

$y^2 = 4px$ or $x^2 = 4py$

$4p = 20$
$p = 5 > 0$

Opens to the right

Vertex: $(0,0)$
Focus: $(5,0)$

Directrix: $x = -5$ (vertical line)

Focal width: $|4p| = |20| = 20$

Endpoints of focal chord: $(5,10)$ & $(5,-10)$
Example 2: Write $6x^2 + 24y = 0$ in standard form and graph it.

\[
6x^2 = -24y
\]

divide by 6

\[
x^2 = -4y
\]

$4p = -4 \Rightarrow p = -1 < 0$

Orientation: opens down

Vertex: $(0, 0)$

Focus: $(0, -1)$ on $y$-axis:

Directrix: $y = 1 (\text{horizontal line})$

Focal width: $|4p| = |-4| = 4$

Endpoints of focal chord: $(-2, -1)$ & $(2, -1)$

$2p = F \rightarrow 2p$
Graphing parabolas with \textbf{vertex not at the origin}:

- Rearrange (complete the square) to look like 
  \[(y - k)^2 = 4p(x - h)\] \quad \text{or} \quad \[(x - h)^2 = 4p(y - k).\]

- Vertex is \((h, k)\). Draw it the same way, except start at this vertex.

\[(x - h)^2 = 4p(y - k)\]

In the above case, the axis of symmetry is the vertical line through the point \((h, k)\), that is \(x = h\).

\[(y - k)^2 = 4p(x - h)\]

\[(y-4)^2 = 12(x-5)\]
\[\vee (5, 4)\]
\[(y + 4)^2 = 12(x+5)\]
\[\vee = (-5, -4)\]

In the above case, the axis of symmetry is the horizontal line through the point \((h, k)\), that is \(y = k\).
Graph of the parabola \((x - h)^2 = 4p(y - k)\).
Example 3: Write $x^2 + 10x = 4y - 1$ in standard form and state the vertex.

Complete the square:

$$x^2 + 10x + 25 = 4y - 1 + 25$$

$$x^2 + 10x + 25 = 4y + 24$$

Standard form:

$$\left(x + 5\right)^2 = 4\left(y + 6\right)$$

Vertex: $(-5, -6)$

$$x - h$$

$$x + 5 = x - (-5) \quad h = -5$$
Example 4: Write \( y^2 - 6y = 8x + 7 \) in standard form and graph it.

\[
y^2 - 6y + 9 = 8x + 7 + 9
\]

\[
\left( y - 3 \right)^2 = 8(x + 2)
\]

- \( (\text{Vertex}) = (0, 3) \)
- \( (\text{Focus}) = (0, 3) \)
- \( (\text{Directrix}) = x = -4 \)
- \( (\text{Focal width}) = 4p = 8, \ p = 2 > 0 \)
- \( (\text{Endpoints of focal chord}) = (1, 1) \ & \ (1, -1) \)
POPPER for Section 8.1:

Question#2: Find the vertex of the parabola \((y + 2)^2 = 4(x - 1)\)?
Example 5: Suppose you know that the vertex of a parabola is at (-3, 5) and its focus is at (1, 5). Write an equation for the parabola in standard form.

\[(y-5)^2 = 16(x+3)\]
Example 6: Suppose you know that the focus of a parabola is \((-1, 3)\) and the directrix is the line \(y = -1\). Write an equation for the parabola in standard form.

\[
(x + h)^2 = 4p(y - k)
\]

\[
(x + 1)^2 = 8(y - 1)
\]
Example 7: Given the following graph, which of the following can be the equation of this parabola?

\[(y-k)^2 = 4\rho (x-h)\]

Try: \((0,0)\)

\((0-4)^2 = 4\rho \cdot (-16)\)

\[16 = 4\rho \cdot (-16)\]

\[4\rho = -1\]

\[(y-4)^2 = -4L(x-16)\]

\[y^2 - 8y + 16 = -x + 16\]

\[-16\]

\[y^2 - 8y = -x\]

\[\Rightarrow x = -y^2 + 8y\]

a) \(x = -y^2 - 4y\)
b) \(x = -y^2 + 4y\)
c) \(x = -y^2 + 8y\)
d) \(x = y^2 + 8y\)
e) \(x = -y^2 - 8y\)
f) None of these
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Question#3: Write this parabola in standard form: \( y^2 - 4y = 4x \) ?

*Hint: Complete the square!*