To form a conic section, we’ll take this double cone and slice it with a plane. When we do this, we’ll get one of several different results.
Part 1 - The Circle

Definition: A circle is the set of all points that are equidistant from a fixed point. The fixed point is called the center and the distance from the center to any point on the circle is called the radius.

An equation of a circle whose center is at the origin will be \( x^2 + y^2 = r^2 \), where \( r \) is the radius of the circle.

For example; \( x^2 + y^2 = 25 \) is an equation of a circle with center \((0, 0)\) and radius 5. Here’s the graph of this circle:
Example 1: State the center and the radius of the circle and then graph it:

\[ x^2 + y^2 = 16. \]

Center: \((0,0)\)

\[ r = 4 \]

\[ x^2 + y^2 = 16 \]

rad. \( r = \sqrt{16} = 4 \)
The standard form of the equation of a circle is 

\[(x-h)^2 + (y-k)^2 = r^2\]

where the center of the circle is the point \((h, k)\) and the radius is \(r\). Notice that if the center of the circle is \((0, 0)\) you’ll get the equation we saw earlier.
\[(x-h)^2 + (y-k)^2 = r^2\]

**Example 2:** State the center and the radius of the circle and then graph it:

\[(x-2)^2 + (y+3)^2 = 16\]

- \(h = 2\)
- \(k = -3\)

\(C(2, -3)\)

Radius = \(\sqrt{4} = 2\)
Popper for Section 8.2:

**Question#1:** Which of the following is the equation of a circle with center at (0,0) and radius 4?

a) \[ 2x^2 + 4y = 1 \]

b) \[ 2x^2 + 2y = 1 \]

c) \[ 2x^2 + 16y = 1 \]

d) \[ 2x^2 - 4y = 1 \]

f) None of these
Sometimes the equation will be given in the general form, and your first step will be to rewrite the equation in the standard form. You’ll need to complete the square to do this.

\(( -\frac{10}{2})^2 = 25\)

**Example 3:** Write the equation in standard form, find the center and the radius and then graph the circle: 

\[ x^2 + y^2 + 6x - 10y + 44 = 26 \]

\[
\begin{align*}
  x^2 + 6x + (6/2)^2 + y^2 - 10y + (10/2)^2 &= -4 + 9 + 25 \\
  (x+3)^2 + (y-5)^2 &= 16
\end{align*}
\]

Center: \((-3, 5)\)

Radius: \(\sqrt{16} = 4\)
**Example 4**: Write the equation in standard form, find the center and the radius and then graph the circle:

\[ 5x^2 + 5y^2 - 20x + 10y = 20 \]

**Divide by 5:**

\[ x^2 + y^2 - 4x + 2y = 4 \]

\[ (x-2)^2 + (y+1)^2 = 9 \]

Center: \((2, -1)\)

Radius: \(\sqrt{9} = 3\)
We can also write the equation of a circle, given appropriate information.

**Example 5:** Write the equation of a circle with center \((2, 5)\) and radius \(2\sqrt{5}\).

\[
(x - h)^2 + (y - k)^2 = \text{radius}^2
\]

\[
(x - 2)^2 + (y - 5)^2 = 20
\]
Example 6: Write an equation of a circle with center \((-1, 3)\) which passes through the point \((4, -7)\).

\[
(x + 1)^2 + (y - 3)^2 = r^2
\]

\[
(x + 1)^2 + (y - 3)^2 = 125
\]

Distance:

\[
distance = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}
\]

\[
= \sqrt{(4 - (-1))^2 + (-7 - 3)^2}
\]

\[
r = \sqrt{25 + 100} = \sqrt{125}
\]

\[
r = \sqrt{125} \Rightarrow r^2 = 125
\]

Or plug in:

\[
(4 + 1)^2 + (-7 - 3)^2 = r^2
\]

\[
5^2 + (-10)^2 = r^2
\]

\[
25 + 100 = r^2
\]

\[
125 = r^2
\]
Example 7: Write an equation of a circle if the endpoints of the diameter of the circle are \((6, -3)\) and \((-4, 7)\).

\[
\text{Center: midpoint}
\]
\[
\text{Center: } \left( \frac{-4+b}{2}, \frac{-3+3}{2} \right)
\]
\[
C(1, 2)
\]
\[
(x-1)^2 + (y-2)^2 = r^2
\]
\[
(6-1)^2 + (-3-2)^2 = r^2
\]
\[
25 + 25 = r^2
\]
\[
50 = r^2
\]
\[
(x-1)^2 + (y-2)^2 = 50
\]
Example 8: What is the equation of the given circle?

Center: $(4, 4)$  
Radius: $r = 5$

Equation of the circle:

$$(x-4)^2 + (y-4)^2 = 5^2$$

$$(x-4)^2 + (y-4)^2 = 25$$
Popper for Section 8.2:

Question#2: What is the center of the circle $x^2 - 4x + y^2 = -3$?
PART 2 – Ellipses

When we slice one of the cones at an angle to the sides of the cone, we get an **ellipse**, as seen in the view from the top (at right).

**Definition:** An ellipse is the set of all points, the sum of whose distances from two fixed points is constant. Each fixed point is called a *focus* (plural = *foci*).
Check this link for an interactive ellipse graph:

Ellipses centered at origin:

Basic "horizontal" ellipse:

Equation: \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b \]

Foci: \((\pm c, 0)\), where \(c^2 = a^2 - b^2\)

Vertices: \((\pm a, 0)\)

Eccentricity: \(e = \frac{c}{a}\)

The eccentricity provides a measure on how much the ellipse deviates from being a circle. The eccentricity \(e\) is a number between 0 and 1.

- small \(e\): graph resembles a circle (foci close together)
- large \(e\): flatter, more elongated (foci far apart)

if the foci are the same, it’s a circle!

Remark: A circle is a special case of an ellipse where \(a=b\).
Basic "vertical" ellipse:

Equation: \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \ a > b \)

Foci: \((0, \pm c)\), where \(c^2 = a^2 - b^2\)

Vertices: \((0, \pm a)\)

Eccentricity: \(e = \frac{c}{a}\)
Graphing ellipses:

To graph an ellipse with center at the origin:

- Rearrange into the form \( \frac{x^2}{\text{number}} + \frac{y^2}{\text{number}} = 1 \).
- Decide if it’s a “horizontal” or “vertical” ellipse.
  - if the bigger number is under \( x^2 \), it’s horizontal (longer in \( x \)-direction).
  - if the bigger number is under \( y^2 \), it’s vertical (longer in \( y \)-direction).
- Use the square root of the number under \( x^2 \) to determine how far to measure in \( x \)-direction.
- Use the square root of the number under \( y^2 \) to determine how far to measure in \( y \)-direction.
- Draw the ellipse with these measurements. Be sure it is smooth with no sharp corners.
- \( c^2 = a^2 - b^2 \) where \( a^2 \) and \( b^2 \) are the denominators. So \( c = \sqrt{\text{big denom} - \text{small denom}} \)
- The foci are located \( c \) units from the center on the long axis.

When graphing, you will need to find the orientation, center, values for \( a \), \( b \) and \( c \), vertices, foci, lengths of the major and minor axes and eccentricity.
**Example 1:** Find all relevant information and graph \( \frac{x^2}{16} + \frac{y^2}{9} = 1 \).

Orientation: **Horizontal**

Center: \((0,0)\)

Vertices: \((-4,0) \& (4,0)\)

\[ c^2 = a^2 - b^2 = 16 - 9 = 7 \]

Foci: \[ c = \sqrt{7} \quad F_1(\sqrt{7},0) \& (-\sqrt{7},0) \]

Length of major axis: \(2a = 2 \cdot 4 = 8\)

Length of minor axis: \(2b = 2 \cdot 3 = 6\)

Coordinates of the major axis: \(V_1(-4,0) \& V_2(4,0)\)

Coordinates of the minor axis: \(C_1(0,3) \& C_2(0,-3)\)

Eccentricity: \(e = \frac{c}{a} = \frac{\sqrt{7}}{4}\)
To graph an ellipse with center not at the origin:

- Rearrange (complete the square if necessary) to look like
  \[
  \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1. \quad \frac{(x-4)^2}{a^2} + \frac{(y-1)^2}{b^2} = 1
  \]

- Start at the center \((h,k)\) and then graph it as before.

When graphing, you will need to find the orientation, center, values for a, b and c, vertices, foci, lengths of the major and minor axes and eccentricity.
Example 2: Find all relevant information and graph \( \frac{(x-1)^2}{9} + \frac{(y+2)^2}{25} = 1 \).

Orientation: \( \sqrt{5} \text{ horizontal} \)

Center: \((1, -2)\)

Vertices: \((1, 3)\) & \((1, -7)\)

\[ c^2 = a^2 - b^2 = 25 - 9 = 16 \Rightarrow c = \sqrt{16} = 4 \]

Foci: \( F_1(1, 2) \) & \( F_2(1, -6) \)

Length of major axis: \( 2a = 2 \cdot 5 = 10 \)

Length of minor axis: \( 2b = 2 \cdot 3 = 6 \)

Eccentricity: \( e = \frac{c}{a} = \sqrt{\frac{4}{5}} \)
Example 3: Write the equation in standard form. Find all relevant information and graph: 

$$4x^2 - 8x + 9y^2 - 54y = -49.$$ 

$$(x - 1)^2 + 9(y - 3)^2 = 36$$

Divide by $36$:

$$\frac{(x-1)^2}{9} + \frac{(y-3)^2}{4} = 1$$

C(1,3)

Horizontal: 

$$a^2 = 9 \Rightarrow a = 3$$

$$b^2 = 4 \Rightarrow b = 2$$

V(1,-2) & (4,3)

$$c^2 = a^2 - b^2 = 9 - 4 = 5$$

$$c = \sqrt{5}$$

$$F_2 = -\sqrt{5}$$

$$F_1 \left(1 + \sqrt{5}, 3\right)$$

$$F_2 \left(1 - \sqrt{5}, 3\right)$$

$$e = \frac{c}{a} = \frac{\sqrt{5}}{3}$$
**Example 4:** Find the equation for the ellipse satisfying the given conditions.

Foci (±3,0), vertices (±5,0)

\[
\begin{align*}
(3,0) & \quad (5,0) \\
(-3,0) & \quad (-5,0)
\end{align*}
\]

\[
\frac{x^2}{25} + \frac{y^2}{16} = 1
\]

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

\[
a = 5 \quad \Rightarrow \quad a^2 = 25
\]

\[
c = 3
\]

\[
c^2 = a^2 - b^2
\]

\[
a = 25 - b^2 \quad \Rightarrow \quad b^2 = 16
\]

\[
b = 4
\]
**Example 5:** Write an equation of the ellipse with vertices (5, 9) and (5, 1) if one of the foci is (5, 7).

**Vertical ellipse**

\[
\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1
\]

- **Center:** midpoint of vertices
  
  \((5, \frac{9+1}{2}) = (5, 5)\)

- **A:** 4, \(a^2 = 16\)

- **C:** 2
  
  \[c^2 = a^2 - b^2\]
  
  \[4 = 16 - b^2 \Rightarrow b^2 = 16 - 4 = 12\]

- **Equation:**
  
  \[
  \frac{(x-5)^2}{12} + \frac{(y-5)^2}{16} = 1
  \]
Remark:
A circle is a special case of an ellipse where \( a = b \):
\[
\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 \quad \rightarrow \quad x^2 + y^2 = a^2
\]
Here, \( c = a^2 - a^2 = 0 \), so the eccentricity is 0.
\[
e = \frac{0}{a} = 0
\]
Popper for Section 8.2:

**Question #3:** Given the ellipse \( \frac{x^2}{9} + \frac{y^2}{25} = 1 \), find the length of the major axis.