MATH 1431 DAY 10

almus@math.uh.edu

Office hours: MWF 11-11:30am, MW: 1-2:15pm

If you e-mail me, please mention your course (1431) in the subject line.

Be considerate of others in class. Respect your friends and do not distract anyone during the lecture.

Bubble in PS ID VERY CAREFULLY! If you make a bubbling mistake, your scantron will not be saved in the system and you will not get credit for it even if you turned it in.

Bubble in Popper Number.

DID YOU RESERVE A SEAT FOR TEST 2?

\[(u \cdot v)' = u' \cdot v + u \cdot v'\]

Popper # 19

**Question # 2**

If \( f(x) = (x^3 + x^2)(4x + 1) \), \( f'(1) = ? \)

a) 10  
b) 18  
c) 33  
d) 29  
e) None

\( f'(x) = u' \cdot v + u \cdot v' = (3x^2 + 2x) \cdot (4x + 1) + (x^3 + x^2) \cdot 4 \)

**Question # 3**

If \( f(x) = \frac{\sqrt{x}}{x^2 + 1} \), \( f'(1) = ? \)

a) 2  
b) 1/2  
c) 1/4  
d) 0  
e) None

\( f'(x) = \frac{1 \cdot (x^2 + 1) - x \cdot (2x)}{(x^2 + 1)^2} \)
Q. #1: For which function can we use Intermediate Value Theorem to prove existence of roots in the given interval?

X I. $f(x) = \frac{x-1}{x}$ , $[-2, 2]$ 

II. $f(x) = x^2 + 4$ , $[-2, 2]$ 

III. $f(x) = x^3 + x$ , $[-2, 2]$ 

f(-2)<0
f(2)>0

a) I and II  b) II only  c) III only  d) II and III  e) None
Recall--

**Theorem: The Product Rule**

If $f$ and $g$ are differentiable at $x$, then so is the product $fg$. Moreover,

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

This formula may be written as:

$$(uv)' = u'v + uv'$$

or

$$\frac{d(u \cdot v)}{dx} = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}.$$
**Theorem: The Chain Rule**

If $g$ is differentiable at $x$ and $f$ is differentiable at $g(x)$, then the composition $f \circ g$ is differentiable at $x$. Moreover,

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

This rule is one of the most important rules of differentiation. It helps us with many complicated functions.

**Example:** Find the derivative of $h(x) = \left(x^2 + \sin x\right)^5$.

$$h'(x) = 5 \cdot (x^2 + \sin x)^4 \cdot (2x + \cos x).$$

**Example:** Find the derivative of $f(x) = \sin^4 x$.

$$f'(x) = 4 \cdot (\sin x)^3 \cdot \cos x = 4 \cdot \sin^3 x \cdot \cos x.$$
\[ y = \cos(x) \quad \Rightarrow \quad y' = -\sin(x) \]

\[ f(x) = \cos(5x) \quad \Rightarrow \quad f'(x) = -\sin(5x) \cdot 5 \]

\[ y = \sin(x) \quad \Rightarrow \quad y' = \cos(x) \]

\[ g(x) = \sin(x^2 + 1) \quad \Rightarrow \quad g'(x) = \cos(x^2 + 1) \cdot 2x \]

\[ f(x) = \]
Example: Find the derivative of \( f(x) = 5 \cos(x^2 + 1) \).

\[
\frac{df}{dx} = \frac{df}{dx} = 5 \cdot \left( -\sin(x^2 + 1) \right) \cdot 2x
\]

\[
= -10x \cdot \sin(x^2 + 1)
\]

Example: Find the derivative of \( h(x) = 2 \sin(x^3) \).

\[
h'(x) = 2 \cdot \cos(x^3) \cdot 3x^2 = 6x^2 \cdot \cos(x^3)
\]
Example: Find the derivative of $g(x) = x^2 \tan(5x)$.

Product rule: 

$$g'(x) = 2x \cdot \tan(5x) + x^2 \cdot \sec^2(5x) \cdot 5$$

Example: Find the derivative of $f(x) = \sin^3(4x)$.

$$f'(x) = 3 \cdot [\sin(4x)]^2 \cdot (\sin(4x))'$$

$$f'(x) = 3 \cdot \sin^2(4x) \cdot \cos(4x) \cdot 4$$

$$= 12 \cdot \sin^2(4x) \cdot \cos(4x)$$

Exercise: Find the derivative of $f(x) = \cos^2(x^2 + 4)$.
Rational Powers

Fact: For \( x \neq 0 \), the derivative of \( f(x) = x^{p/q} \) is:

\[
\frac{df}{dx} = \frac{p}{q} x^{(p/q)-1}.
\]

Furthermore, if \( u \) is a differentiable function at \( x \) and \( f(x) = u^{p/q} \), then using the chain rule:

\[
\frac{df}{dx} = \frac{p}{q} u^{(p/q)-1} \cdot \frac{du}{dx}.
\]

Example: Find the derivative of \( f(x) = \sqrt{5x+7} \).

\[
f'(x) = \frac{1}{2\sqrt{5x+7}} \cdot 5 = \frac{5}{2\sqrt{5x+7}}.
\]
Example: Find the derivative of $f(x) = \sqrt{x^3 + x}$.

$$f'(x) = \frac{1}{2 \cdot \sqrt{x^3 + x}} \cdot (3x^2 + 1)$$

Example: Find the derivative of $f(x) = x^{1/3} + \sqrt[5]{2}/5$.

$$f'(x) = \frac{1}{3} \cdot x^{1/3 - 1} + \frac{2}{5} \cdot x^{2/5 - 1} = \frac{1}{3} \cdot x^{-2/3} + \frac{2}{5} \cdot x^{-3/5}$$

Example: Find the derivative of $f(x) = x(2x + 1)^{1/3}$.

$$f'(x) = 1 \cdot (2x + 1)^{1/3} + x \cdot \frac{1}{3} \cdot (2x + 1)^{-2/3} \cdot 2$$
Example: $f(x) = \sin(x^2)$  \[ f''(x) = ? \]

\[ f'(x) = 2x \cdot \cos(x^2) \]

\[ f'''(x) = 2 \cdot \cos(x^2) + 2x \cdot -\sin(x^2) \cdot 2x \]
The Chain Rule in Leibniz Notation

This is what the chain rule says with Leibniz’s double-$d$ notation:

\[
\frac{d}{dx} \left[ f(u(x)) \right] = f'(u(x)) \cdot u'(x) \quad \text{or} \quad \frac{d}{dx} \left[ f(u) \right] = f'(u) \cdot \frac{du}{dx}.
\]

If \( y = f(u) \), then

\[
\frac{dy}{dx} = f'(u) \cdot \frac{du}{dx}
\]

and since \( f'(u) = \frac{dy}{du} \), the chain rule can be written as:

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.
\]

That is, the derivative of \( y \) with respect to \( x \) is the product of the derivative of \( y \) with respect to \( u \) and the derivative of \( u \) with respect to \( x \).

This formula can be extended to more variables; each new variable adds a new link to the chain.

For the composition of 3 functions,

\[
\frac{d}{dx} \left[ f(u(v(x))) \right] = f'(u(v(x))) \cdot u'(v(x)) \cdot v'(x)
\]

can be written as:

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}.
\]
Example: If $y = u^2 + 1$, $u = 5x - 1$ and $x = 3t^2$, find $\frac{dy}{dt}$.

$\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dx} \cdot \frac{dx}{dt}$

$\left. \frac{dy}{dt} \right|_{t=1} = (2u) \cdot 5 \cdot (6t) \bigg|_{t=1} = (2 \cdot 14) \cdot 5 \cdot (6 \cdot 1)$

$\frac{dy}{dt} \bigg|_{t=1} = 840$

$(f \circ g)' = f'(g) \cdot g'$

IMPORTANT Example: The following information is given about two functions $f$ and $g$.

$f(1) = 6$, $f'(1) = 4$, $f(7) = 2$, $f'(7) = 1$,

$g(1) = 7$, $g'(1) = 8$, $g(6) = 10$, $g'(6) = 2$.

a) If $h(x) = (f \circ g)(x)$, find $h'(1)$.

$h'(x) = f'(g(x)) \cdot g'(x)$

$h'(1) = f'(g(1)) \cdot g'(1)$

$h'(1) = f'(7) \cdot 8 = 1 \cdot 8 = 8$
b) If \( h(x) = (fg)(x) \), find \( h'(1) \).

\[
h'(x) = f'(x)g(x) + f(x)g'(x)
\]

\[
h'(1) = f'(1)g(1) + f(1)g'(1)
\]

\[
= 4 \cdot 7 + 6 \cdot 8 = 28 + 48
\]

\[
= 76
\]

c) If \( h(x) = \left( \frac{f}{g} \right)(x) \), find \( h'(1) \).

\[
h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}
\]

\[
h'(1) = \frac{f'(1)g(1) - f(1)g'(1)}{[g(1)]^2} = \frac{4 \cdot 7 - 6 \cdot 8}{[7]^2}
\]

\[
= \frac{28 - 48}{49} = -\frac{20}{49}
\]

Exercises:

d) If \( h(x) = [f(x)]^3 \), find \( h'(1) \).

e) If \( h(x) = (g \circ f)(x) \), find \( h'(1) \).
Popper #

Question#\[
y = 4 cos^2(x), \quad \frac{dy}{dx} \bigg|_{x=\frac{\pi}{6}} = ?
\]

a) $-2\sqrt{3}$
b) $2\sqrt{3}$
c) $4\sqrt{3}$
d) $-4\sqrt{3}$
e) None

\[\boxed{0.5}\]

4 - 5: Mark (A)