Math 1431 DAY 14

BBBULE IN PS ID VERY CAREFULLY! If you make a bubbling mistake, your scantron will not be saved in the system and you will not get credit for it even if you turned it in.

Be considerate of others in class. Respect your friends and do not distract anyone during the lecture.

Section 3.1

Related Rates:

- Draw a “picture”.
- What do you know?
- What do you need to find?
- Write an equation involving the variables whose rates of change either are given or are to be determined. (This is an equation that relates the parts of the problem.)
- Implicitly differentiate both sides of the equation with respect to time. This FREEZES the problem.
- Solve for what you need.

Example 6: A 6 foot man is walking towards a 25 foot lamp post at the rate of 10 feet per second. How fast is the length of his shadow changing when he is 20 feet from the lamp post?

\[
\frac{dx}{dt} = -10 \text{ ft/sec}
\]

\[
\frac{dy}{dt} \bigg|_{x=20} = ?
\]
\[
\frac{25}{x+y} \times \frac{6}{y}
\]

\[
25y = 6x + 6y
\]

\[-by \quad -6y\]

19. \( y = 6x \)

Deriv. w. respect to time

\[
19. \frac{dy}{dt} = 6 \cdot \frac{dx}{dt}
\]

\[
\frac{dy}{dt} = \frac{-60}{19} \quad \text{ft/sec}
\]
Example 7: If a rocket is rising vertically at the rate of 1200 ft/sec when it is 4000 feet up, how fast is the camera-to-rocket distance changing at the instant?

\[ \frac{dy}{dt} = 1200 \text{ ft/sec} \]

\[ \frac{dz}{dt} \bigg|_{y=4000} = ? \]

\[ y^2 + 3000^2 = z^2 \]

\[ 2y \cdot \frac{dy}{dt} + 0 = 2z \cdot \frac{dz}{dt} \]

\[ 4000 \cdot 1200 = 5000 \cdot \frac{dz}{dt} \]

\[ \frac{dz}{dt} = \frac{4000 \cdot 1200}{5000} = \frac{4800}{5} = 960 \text{ ft/sec} \]
Example 8: Using the same conditions for the rocket in #7, how fast must the camera elevation angle change at the instant to keep the rocket in sight?

\[ \frac{dy}{dt} = 1200 \text{ ft/sec} \]

\[ \frac{d\theta}{dt} \bigg|_{y=4000} = ? \]

\[ \tan\theta = \frac{y}{3000} = \frac{1}{3000} \cdot y \]

\[ \sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{3000} \cdot \frac{dy}{dt} \]

\[ = \frac{1200}{3000} = \frac{1}{25} \]

\[ \frac{9}{25} \cdot \left( \frac{5}{3} \right)^2 \cdot \frac{d\theta}{dt} = \frac{1}{3000} \cdot 1200 \cdot \frac{9}{25} \]

\[ \frac{d\theta}{dt} = \frac{1}{3000} \cdot 1200 \cdot \frac{9}{25} \text{ rad/sec} \]
A rocket is rising vertically at the rate of 1000 ft/sec

**Question #** How fast is the camera-to-rocket distance changing when it is 3000 feet up?

a) $\frac{3000}{5}$
b) $\frac{4000}{5}$
c) $\frac{5000}{3}$
d) $\frac{5000}{12}$
e) None

**Question #** How fast must the camera elevation angle change at the instant when height is 3000 feet to keep the rocket in sight?

a) $\frac{4}{25}$
b) $\frac{16}{25}$
c) $\frac{25}{16}$
d) $\frac{25}{4}$
e) None
\[ s(t) = t^2 + 1 \]

\[ t = 5 \quad s(5) = 26 \]

\[ \omega(t) : \text{velocity} = s'(t) \]

\[ a(t) : \text{acceleration} = \omega'(t) \]
Position, Velocity, Acceleration

Velocity

Imagine a particle moving along a straight-line path in some way. On this line, choose a point of reference, a positive direction and a negative direction. For example, we can choose the line to be the $x$–axis, reference point can be the origin, moving to the right is the positive direction, and left is the negative direction.

Let $t$ be a variable representing the time elapsed since some reference time (when $t = 0$). Let $s(t)$ be the position of the particle at time $t$ measured relative to some reference point (where $s = 0$). If the position function $s(t)$ is differentiable, then the derivative $s'(t)$ gives the rate of change of the position function at time $t$.

This rate is called the velocity at time $t$ and denoted as $v(t)$. In symbols,

$$v(t) = s'(t).$$

Average velocity over a time interval $a \leq t \leq b$ is the change in position divided by the change in time:

$$v_{avg} = \frac{\Delta s}{\Delta t} = \frac{s(b) - s(a)}{b - a}.$$

Acceleration

Acceleration is defined to be the rate of change of velocity per unit time. If the velocity function $v(t)$ is differentiable, then its derivative gives the acceleration function;

$$a(t) = v'(t).$$

Since, $v(t)$ is the derivative of $s(t)$, acceleration is the second derivative of position:

$$a(t) = v'(t) = s''(t) \quad \text{or} \quad a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}.$$
Here are the connections between position, velocity, acceleration and speed:

- Positive velocity corresponds to motion in positive direction (position is increasing). Negative velocity corresponds to motion in negative direction (position is decreasing).

- Positive acceleration corresponds to increasing velocity. Negative acceleration corresponds to decreasing velocity.

- The object is **speeding up** if the velocity and acceleration have the same sign. The object is **slowing down** if the velocity and acceleration have opposite signs.

\[
\text{speed} = |\text{velocity}|
\]

\[
\omega(t) \quad \& \quad a(t)
\]

\[
+ \quad - \quad + \quad -
\]
**Example:** An object moves along the $x$-axis and its position is given by the function

$$s(t) = t^3 - 9t^2 + 15t + 8.$$ 

a) Find the velocity of this object at time $t = 2$.

$$v(t) = s'(t) = 3t^2 - 18t + 15 \quad v(2) = 12 - 36 + 15 = -9$$

b) Find the acceleration at time $t = 2$.

$$a(t) = v'(t) = 6t - 18 \quad a(2) = 12 - 18 = -6$$

c) When does this object change direction?

$$v(t) = 3t^2 - 18t + 15 = 3(t^2 - 6t + 5) = 3(t - 5)(t - 1)$$

Stop at $t = 5$ and $t = 1$

$[t = 1 \& t = 5]$  

$d)$ When is the object speeding up?

$(1, 3) \& (5, \infty)$

e) When is it slowing down?

$$a(t) = 6t - 18 = 0 \quad t = 3$$

$(0, 1) \& (3, 5)$

Should know how to make a “sign chart” to solve inequalities!  
We will use it a lot in this chapter.
Exercise: If \( x(t) = \frac{1}{2} t^4 - 5t^3 + 12t^2 \), find the velocity of the moving object when its acceleration is zero.

\[
v(t) = x'(t) = 2t^3 - 15t^2 + 24t
\]

\[
a(t) = 6t^2 - 30t + 24 = 0 \Rightarrow 6(t^2 - 5t + 4) = 0
\]

\[
6(t - 4)(t - 1) = 0
\]

\[
t = 4 \quad \text{or} \quad t = 1.
\]

\[
\begin{align*}
v(1) &= 2 - 15 + 24 = 11 \\
v(4) &= 2 \cdot 4^3 - 15 \cdot 4^2 + 24 \cdot 4 = \ldots
\end{align*}
\]

**Free Fall Formulas**

The height of an object in free fall is given by

\[
h(t) = -16t^2 + v_0t + h_0 \quad \text{(distance in feet)}
\]

or

\[
h(t) = -4.9t^2 + v_0t + h_0 \quad \text{(distance in meters)},
\]

where \( v_0 \) is the initial velocity and \( h_0 \) is the initial height.
**Example:** An object is dropped from a height of 320 feet. \( v_0 = 0 \quad h_0 = 320 \)

a) What is its height after 1 second? \( h(t) = -16t^2 + 320 \)
\[ h(1) = -16(1)^2 + 320 = 304 \text{ ft} \]

b) What is its velocity at time \( t=1 \)? \( \omega(t) = h'(t) = -32t \)
\[ \omega(1) = -32 \]

c) How long does it take for the object to hit the ground?
\[ -16t^2 + 320 = 0 \]
\[ 16t^2 = 320 \quad t^2 = 20 \quad t = \pm \sqrt{20} \]
\[ t = 2\sqrt{5} \]

d) What is the speed on impact?
\[ \omega(t) = -32t \]
\[ \omega(2\sqrt{5}) = -32 \cdot 2\sqrt{5} = -64\sqrt{5} \]

**Speed:** \[ | -64\sqrt{5} | = 64\sqrt{5} \]
**Example:** A stone, projected upward with an initial velocity of 112 ft/sec, moves according to \( x(t) = -16t^2 + 112t \).

a) Compute the velocity and acceleration when \( t = 3 \) and when \( t = 4 \).

\[
\begin{align*}
  v(t) &= -32t + 112 \\
  v(3) &= -96 + 112 = 16 \\
  v(4) &= -32 \cdot 4 + 112 = 16 \\
  a(t) &= -32 \\
  a(3) &= -32 \\
  a(4) &= -32.
\end{align*}
\]

\[
\text{max height}
\]

b) Determine the greatest height the stone will reach.

\[
\text{Solve } v(t) = 0 \Rightarrow -32t + 112 = 0 \Rightarrow t = \frac{112}{32} = \frac{7}{2} \text{ sec}.
\]

\[
\text{max height at } t = \frac{7}{2} \Rightarrow h\left(\frac{7}{2}\right) = -16 \cdot \left(\frac{7}{2}\right)^2 + 112 \cdot \frac{7}{2}
\]

\[
h_{\text{max}} = 196 \text{ ft}.
\]

c) Determine when the stone will have a height of 96 ft.

\[
\text{Solve } h(t) = 96
\]

\[
-16t^2 + 112t = 96
\]

\[
\Rightarrow -16t^2 + 112t - 96 = 0
\]

\[
-16 (t^2 - 7t + 9) = 0
\]

\[
(t - 8)(t - 1) = 0
\]

\[
\Rightarrow t = 8 \text{ or } t = 1.
\]
**Exercise:** Supplies are dropped from a stationary helicopter and seconds later hit the ground at 98 meters per second. How high was the helicopter?

\[
\text{speed on impact } = 98 \quad \Rightarrow \quad v(t) = -98 \quad \text{m/s} \\
\begin{align*}
\text{h}(t) & = -4.9 \quad t^2 + h_0 \\
v(t) & = -9.8 \quad t
\end{align*}
\]

Now use \( h(10) = 0 \)

\[ -4.9 \cdot (10)^2 + h_0 = 0 \]

\[ h_0 = 490 \quad \text{ft} \]

**EXTRA Examples (Exercises for you!)**

Ex: \( y(t) = -16 \quad t^2 + y_0 \). The object hits the ground in 4 seconds. What is the initial height?

Ex: \( y(t) = -16 \quad t^2 + 80 \quad t \).

a) How long does it take to reach the maximum height? What is the maximum height?

b) When does the object reach 64 feet?

c) What is the speed of this object when the height is 64 feet?

Ex: \( y(t) = -16 \quad t^2 + 160t \).

a) What is the height of this object after 2 seconds?

b) What is the velocity after 2 seconds?

c) When does the object reach velocity 80 ft/sec?

d) When does the object reach height 80 ft?

e) What is the speed of this object when the height is 144 ft.?

f) When does the object hit the ground?

g) What is the speed on impact when it hits the ground?