Review for Test 3

Material: Chapter 3

Time: 50 minutes

Number of questions: 10 + bonus

MC: 7 (8 pts each; 56 pts)

FR: 3 questions

The grade you’ll see right away will be for MC part only (out of ?? points).

For related rates question: There will be two options; solve only one of them. If you solve both, ONLY the first one will be graded.

Do not be late for your test. If you’re more than 10 minutes late, CASA will not let you in.

If you miss your test, try to reschedule. If you can’t, you will get a 0. Final replaces one missed test (or your lowest test – if you haven’t missed any tests).

Take practice test 3! 5% of your best score will be added to your test grade.

How to study? Go over class notes, rework past quizzes, emcfs and poppers. Work on the review sheet posted on my website and take the practice test.

What to know:

- Related Rates
- How to determine the intervals of increase/decrease, concave up/down.
- Critical points, local min/max, absolute min/max, points of inflection.
- First derivative test, Second derivative test.
- Given the graph of f’, what can you conclude about f? 
- Given the graph of f”, what can you conclude about f? 
- Finding vertical tangents, cusps.
- Graphing a function
Find the critical numbers, if any, and the intervals on which the function is increasing or decreasing. **Classify** the critical points.

1. \( f(x) = x^3 + 3x^2 - 9x + 1 \)
   
   \[ f'(x) = 3x^2 + 6x - 9 = 0 \]
   
   \[ 3(x^2 + 2x - 3) = 0 \]
   
   \[ 3(x + 3)(x - 1) = 0 \]
   
   **CP:** \( x = -3 \), \( x = 1 \)

2. \( f(x) = 4\sqrt[3]{x^2 - 6x} \)
   
   \[ f'(x) = 4 \cdot \frac{1}{3} \cdot (x^2 - 6x)^{-2/3} \cdot (2x - 6) \]
   
   \[ = \frac{4(2x - 6)}{3(x^2 - 6x)^{2/3}} = \frac{8(x - 3)}{3(x, (x - 6))^{2/3}} \]

   **CP:** \( x = 3 \), \( x = 9 \), \( x = 6 \)

   Vertical tangent at \( 0 \)
   Local min at \( 3 \)
   Vertical tangent at \( 6 \)
\[ f(x) = (x - 4)^{\frac{1}{5}} \]

\[ f'(x) = \frac{1}{5(x - 4)^{\frac{4}{5}}} \]
Exercise: Find critical points and classify them:

\( f(x) = \tan(2x) - 8x \) on \([0, \pi/2)\).

4. Find all local and absolute extrema (if any) of \( f \) on the given intervals and state where those values occur.

\[ f(x) = 3x^2 - 6x - 4 \quad [-1, 3] \]

\[ f'(x) = 6x - 6 = 0 \quad \Rightarrow x = 1 \]

\[ f''(x) = 6 \quad \Rightarrow f''(1) = 6 > 0 \quad \text{local min at } x = 1. \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>[ f(-1) = 3 - 6 - 4 = -5 ] abs max at ((-1, 5)) and ((2, 5))</td>
</tr>
<tr>
<td>1</td>
<td>[ f(1) = 3 - 6 - 4 = -7 ]</td>
</tr>
<tr>
<td>3</td>
<td>[ f(3) = 27 - 18 - 4 = 5 ] abs min at ((1, -7)).</td>
</tr>
</tbody>
</table>
5. Given \( f(x) = \frac{1}{20} x^5 - \frac{1}{4} x^4 + 2 \), when is \( f \) concave up? Locate points of inflection, if any.

\[
f'(x) = \frac{1}{4} x^4 - x^2
\]

\[
f''(x) = x^3 - 3x^2 = 0
\]

Possible roots are \( x = 0 \) and \( x = 3 \). The sign chart shows

- \( f''(x) > 0 \) for \( x > 3 \),
- \( f''(x) < 0 \) for \( 0 < x < 3 \),
- \( f''(x) > 0 \) for \( x < 0 \).

\( f \) concaves up on \((3, \infty)\).

6. Given that a particle moves with \( s(t) = t^3 - 6t^2 + 12t + 1 \). Find the velocity at \( t = 1 \).

Find the acceleration at \( t = 1 \).

\[
\begin{align*}
\text{Velocity:} & \quad v(t) = s'(t) = 3t^2 - 12t + 12 \\
\text{Velocity at } t = 1: & \quad v(1) = 3 - 12 + 12 = 3 \\
\text{Acceleration:} & \quad a(t) = v'(t) = 6t - 12 \\
\text{Acceleration at } t = 1: & \quad a(1) = 6 \cdot 1 - 12 = -6
\end{align*}
\]
7. Verify that Mean Value Theorem applies. Find the \( c \) satisfying the theorem.

\[
f(x) = x^3 - x \quad \text{over} \quad [1,3].
\]

1. \( f \) is continuous on \([1,3]\) \( \checkmark \)

2. \( f \) is differentiable on \((1,3)\) \( \checkmark \)

By MVT, there is at least one \( c \) in \((1,3)\) such that

\[
f'(c) = \frac{f(3) - f(1)}{3 - 1} = \frac{24 - 0}{2} = 12
\]

\[
\begin{cases} 
    f'(c) = 3c^2 - 1 = 12 \\
    \Rightarrow 
    3c^2 = 13 \\
    c^2 = \frac{13}{3} \\
    c = \pm \sqrt{\frac{13}{3}}
\end{cases}
\]

\[\Rightarrow \quad c = \sqrt{\frac{13}{3}}\]
8. The graph of $f'(x)$ is given.

When is $f(x)$ increasing? $(-\infty, 2) \cup (4, \infty)$

When is $f(x)$ decreasing? $(2, 4)$

When is $f(x)$ concave up? $(-\infty, 1) \cup (3, \infty)$

When is $f(x)$ concave down? $(-\infty, -1) \cup (1, 3)$

Critical numbers of $f(x)$:

$x = -1$ \quad $x = 2$ \quad $x = 4$

Points of inflection for $f(x)$:
9. The graph of $f''(x)$ is given.

When is $f(x)$ concave up?

When is $f(x)$ concave down?

Points of inflection for $f(x)$:

\[ [-2, -1), (-1, 2], (2, \infty) \]
10. State the domain, any asymptotes, critical numbers, the intervals where the function is increasing/decreasing, concave up/down, and GRAPH this function.

\[ f(x) = \frac{-10x}{(x-4)^2} \]

- Domain: \( x \neq 4 \)
- \( V.A.: x = 4 \)
- \( H.A.: y = 0 \)
- Critical Numbers: \( f'(x) = \frac{10(x+4)}{(x-4)^3} \)

\[ f''(x) = \frac{-20(x+6)}{(x-4)^4} \]

- \( f \) increases on \((-\infty, -6) \cup (4, \infty)\)
- \( f \) decreases on \((-6, 4)\)
- Local max at \((-4, \frac{5}{9})\)

Graph showing the function with key points and behaviors indicated.
EXERCISE: State the domain, any asymptotes, critical numbers, the intervals where the function is increasing/decreasing, concave up/down, and GRAPH this function.

\[ f(x) = \frac{x^2}{(x + 2)^2} \]

\[ f'(x) = \frac{4x}{(x + 2)^3} \]

\[ f''(x) = \frac{-8x + 8}{(x + 2)^4} \]
EXERCISE:

What can you say about a function with these properties:

1. The domain is all real numbers except 3 and -3.
2. The function has vertical asymptotes at $x = 3$ and $x = -3$.
3. The function is symmetric about the y-axis.
4. $\lim_{x \to \infty} f(x) = -1$
5. $f(0) = 0, f(2) = 0, f(4) = 0$
6. $f'(x) < 0$ for $0 < x < 1$ and $x > 3$
7. $f'(x) > 0$ for $1 < x < 3$
8. $f''(x) < 0$ for $0 < x < 1/2$
Exercise: State the domain, any asymptotes, critical numbers, the intervals where the function is increasing/decreasing, concave up/down, and GRAPH this function.

\[ f(x) = -3x^4 + 12x^3 - 20 \]
Related Rates

- Work all the problems in your homework and quizzes (solve: shadow length, ladder, area of a rectangle, area of a triangle, area of a circle, volume of a sphere, volume of a cone, etc.)
- Must know basic area and volume formulas from geometry.
  Area: Circle, Square, Triangle, Rectangle
  Surface area: Cube, Rectangular Prism
  Volume: Cube, Cone, Cylinder, Sphere, Rectangular Prism

12. On a morning when the sun will pass directly overhead, the shadow of a 60-ft tower on level ground is 45 feet long. At the moment in question, the angle the sun makes with the ground is increasing at the rate of $\frac{\pi}{600}$ radians/minute. Find the rate of change of the shadow's length.

\[
\tan \theta = \frac{60}{x} = 60 \cdot x^{-1}
\]

\[
\sec^2 \theta \cdot \frac{d\theta}{dt} = -60 \cdot x^{-2} \cdot \frac{dx}{dt}
\]

\[
\frac{45^2}{-60} \cdot \left(\frac{5}{3}\right)^2 \cdot \frac{11}{600} = -60 \cdot \frac{1}{45^2} \cdot \frac{dx}{dt} \cdot \frac{45^2}{-60}
\]

\[
\frac{45^2}{-60} \cdot \left(\frac{5}{3}\right)^2 \cdot \frac{11}{600} = -60 \cdot \frac{1}{45^2} \cdot \frac{dx}{dt} \cdot \frac{45^2}{-60}
\]
\[
\Rightarrow \quad \frac{dx}{dt} = \left( \frac{5}{3} \right)^2 \cdot \frac{\pi}{600} \cdot \frac{45^2}{-60} \\
\frac{dx}{dt} = \frac{25}{9} \cdot \frac{\pi}{600} \cdot \frac{45^2}{-60} \quad \cdots
\]
13. Water is poured into a conical tank at the rate of 64 cubic feet per minute. If the tank is 10 feet tall and the top of the tank has a radius of 6 feet, how fast does the radius of the water surface change when the water is 2 feet deep?
14. Two cars are moving towards the same point. The first car started from a point that is 100 miles away and it travels south at a rate of 40 mi/h. The second one started from a point 105 miles away and it travels east at a rate of 60 mi/h. At what rate is the distance between the cars changing one hour later?

\[
\begin{align*}
\frac{dx}{dt} &= -40 \text{ mi/h} \\
\frac{dy}{dt} &= -60 \text{ mi/h} \\
\frac{dz}{dt} | _{t=1} &= ? \\

2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} &= 2z \cdot \frac{dz}{dt} \\
\end{align*}
\]

\[
\begin{align*}
x &= 60 \\
y &= 45 \\
z &= 75 \\
\end{align*}
\]

\[
\frac{dz}{dt} = \frac{-2400 - 1500}{75} = \frac{-3900}{75} \text{ mi/h} \\
\text{reduce...}
\]
12. The length of a rectangle is increasing at a rate of 5 in/min and the width is decreasing at a rate of 1 in/sec. At what rate is the area changing when the width is 5 inches and the length is twice the width?

\[ \frac{dx}{dt} = 5 \quad \frac{dy}{dt} = -1 \]

\[ A = x \cdot y \]

\[ \frac{dA}{dt} = \frac{dx}{dt} \cdot y + x \cdot \frac{dy}{dt} \]

\[ \frac{dA}{dt} = 5 \cdot 5 + 10 \cdot (-1) \]

\[ = 25 - 10 = 15 \text{ in}^2/\text{sec} \]
The altitude of a triangle is increasing at a rate of 5 in/min while the base is decreasing at a rate of 1 in/min. At what rate is the area of the triangle changing when the altitude is 10 in and the base is 6 in?

\[
A = \frac{1}{2} \cdot b \cdot h
\]

\[
\frac{dA}{dt} = \frac{1}{2} \cdot \frac{db}{dt} \cdot h + \frac{1}{2} \cdot b \cdot \frac{dh}{dt}
\]

\[
= \frac{1}{2} \cdot (-1) \cdot 10 + \frac{1}{2} \cdot 6 \cdot 5
\]

\[
\frac{dA}{dt} = -5 + 15 = 10 \ \text{in}^2/\text{min}
\]
**POPPER#**

**Question#** Is there a vertical tangent or a vertical cusp at the critical number for 
\[ f(x) = (x + 3)^{\frac{2}{3}} \] 

a. Vertical tangent  b. cusp

**Question#** Is there a vertical tangent or a vertical cusp at the critical number for 
\[ f(x) = (x - 5)^{\frac{1}{5}} \] 

a. Vertical tangent  b. cusp

**Question#** Classify \( x = -1 \) if the graph is the derivative of \( f \).

a. relative maximum  
b. relative minimum  
c. POI  
d. who cares?
Find the POI of $f$ given the graph of $f'(x)$.

a. -1, 2, 4
b. 0, 2
c. 0, 2, 3.5
d. who cares?