Chapter 5 Applications of Derivatives

Section 5.1 – Optimization

Optimization problems (to maximize or minimize):

1. Draw a picture, label it.
2. Determine the primary function (what is to be a max/min)
3. Use a secondary formula if necessary to get the primary function in terms of one variable.
4. Determine a feasible domain.
5. Find the max/min.
6. SHOW that the answer is a max/min using the First or Second Derivative test.
To maximize/minimize a function on a closed bounded interval, we evaluate the function at the endpoints, and then evaluate the function at any critical numbers in the interval.

**Example 3:** A rectangle sits in the first quadrant with its base on the x-axis and its left side on the y-axis. Its upper right hand corner is on the line passing through the points \((0, 4)\) and \((3, 0)\). What is the largest possible area of this rectangle?

The equation of the line is:

\[ y = -\frac{4}{3}x + 4 \]

The area of the rectangle is:

\[ A = x \cdot y = x \left(-\frac{4}{3}x + 4\right) \]

\[ A = -\frac{4}{3}x^2 + 4x \]

To find the maximum area, we differentiate the area with respect to \(x\):

\[ A' = -\frac{8}{3}x + 4 \]

Setting the derivative equal to zero gives:

\[ -\frac{8}{3}x + 4 = 0 \]

Solving for \(x\):

\[ x = -\frac{3}{4} \cdot \frac{3}{8} = \frac{3}{2} \]

The second derivative test confirms this is a maximum:

\[ A'' = -\frac{8}{3} \]

\[ A'' \left(\frac{3}{2}\right) = -\frac{8}{3} < 0 \]

\(x = \frac{3}{2}\) is a max. 

\(\checkmark\)
Largest area: \( x \cdot y = x \cdot \left( -\frac{4}{3}x + 4 \right) \)

Plug in \( x = \frac{3}{2} \)

\[
= \frac{3}{2} \cdot \left( -\frac{4}{3} \cdot \frac{3}{2} + 4 \right)
\]

\[
= \frac{3}{2} \cdot 2 = 3
\]

Dimensions: \( x \) by \( y \)

\[
\frac{3}{2} \text{ by } y \quad -\frac{4}{3} \cdot \frac{3}{2} + 4 = 2
\]
Example 4: Find the largest possible area for a rectangle with base on the x-axis and upper vertices on the curve $y = 4 - x^2$.

Let $A = 2x y$ to be max.

$$A = 2x (4 - x^2)$$

$$A = 8x - 2x^3$$

To be max.

$$\Rightarrow 8 = 6x^2 \Rightarrow x^2 = \frac{8}{6} = \frac{4}{3}$$

$$\Rightarrow x = \pm \frac{2}{\sqrt{3}}$$

Critical point: $x = \frac{2}{\sqrt{3}}$ ($x > 0$)

Second derivative test: $A'' = -12x$.

$$A'' \left( \frac{2}{\sqrt{3}} \right) = -12 \cdot \frac{2}{\sqrt{3}} < 0$$

Max at $x = \frac{2}{\sqrt{3}}$

$$A_{\text{max}} = 8x - 2x^3$$

Plug $x = \frac{2}{\sqrt{3}}$:

$$= 8 \cdot \frac{2}{\sqrt{3}} - 2 \cdot \left( \frac{2}{\sqrt{3}} \right)^3 = \frac{16}{\sqrt{3}} - \frac{16}{3 \sqrt{3}} \cdot \frac{3}{3}$$

$$= \frac{16 \sqrt{3}}{3} - \frac{16 \sqrt{3}}{9} = \frac{32 \sqrt{3}}{9}$$
Example 5: Square corners are cut from a rectangular piece of tin that is 24 cm by 45 cm. The edges are folded up to form an open box. Find the length of the side of the square corner removed in order to have a box with a maximum volume.

\[ V = \text{base area} \cdot \text{height} = (45-2x)(24-2x) \cdot x \]

\[ V = (45-2x)(24x-2x^2) \]

\[ V' = -2 \cdot (24x-2x^2) + (45-2x) \cdot (24-4x) \]

\[ V' = 12x^2 - 276x + 4524 \]

\[ V' = 12(x^2-23x+90) = 12(x-18)(x-5) \]

\[ V' = 0 \quad \text{if} \quad x = 18 \quad \text{or} \quad x = 5. \]

\[ \text{cp:} \quad x = 5 \]
2nd der. test: \( V'' = 24x - 276 \) \( V''(5) = 24 \cdot 5 - 276 < 0 \)

max at \( x = 5 \) \( \checkmark \)

To maximize volume; \( x = 5 \).
Example 6: Find the point(s) on the graph of $y = 4 - x^2$ closest to $(0, 2)$.

$d = \sqrt{(x-0)^2 + (y-2)^2}$

to be minimized.

\[
d = \sqrt{x^2 + (4-x^2-2)^2}
\]
\[
d = \sqrt{x^2 + (2-x^2)^2}
\]

$D = x^2 + (2-x^2)^2$; minimize $D$ instead.

\[
D' = 2x + 2(2-x^2)(-2x) = 0
\]

\[
2x - 8x + 4x^3 = 0
\]

\[
4x^3 - 6x = 0 \Rightarrow 2x(2x^2 - 3) = 0
\]

\[
\Rightarrow x = 0 \quad x = \pm \sqrt{\frac{3}{2}}
\]

3 critical points!
$2^{nd}$ deriv. test: $D'' = 12x^2 - 6$

$x = 0 \Rightarrow D''(0) = -6 < 0 \Rightarrow \text{max} \times \times$

$x = \frac{\sqrt{3}}{2} \Rightarrow D''\left(\frac{\sqrt{3}}{2}\right) = 12 \cdot \frac{3}{2} - 6 > 0 \Rightarrow \min \downarrow$

$x = -\frac{\sqrt{3}}{2} \Rightarrow D'' = 12 \cdot \frac{3}{2} - 6 > 0 \Rightarrow \min \downarrow$

$y = 4-v^2$

Closest points:

\[
\left(\frac{\sqrt{3}}{2}, 4 - \frac{3}{2}\right) \& \left(-\frac{\sqrt{3}}{2}, 4 - \frac{3}{2}\right)
\]

\[
\left(\frac{\sqrt{3}}{2}, \frac{5}{2}\right) \& \left(-\frac{\sqrt{3}}{2}, \frac{5}{2}\right)
\]

Work on Quil 21
EMCF 10 & HW 10

#24: 1-5; B

* To find minimum distance: $d = \sqrt{\ldots}$ plug $x = \frac{\sqrt{3}}{2}$
Example 7: An open top box with a square base is to be built to hold 32 cubic feet. What should the dimensions be in order to minimize the cost of material used to build this box?

\[ V = 32 \text{ ft}^3 \]

\[ x^2 \cdot y = 32 \]

\[ y = \frac{32}{x^2} \]

to be minimized

\[ SA: x^2 + 4xy \]

\[ \Rightarrow \]

\[ \text{Minimum} \]

\[ SA: x^2 + 4x \cdot \frac{32}{x^2} = x^2 + \frac{128}{x} \]

\[ \Rightarrow \]

\[ 2x^2 = 128 \]

\[ \Rightarrow \]

\[ x^2 = 64 \]

\[ x = \sqrt[3]{64} = 4 \]

\[ \text{CP: } x = 4 \]

2nd deriv. test:

\[ SA'' = 2 + \frac{256}{x^3} \]

\[ SA'' (4) > 0 \text{ min at } x = 4 \checkmark \]

Dimensions are:

\[ x \text{ ft by } x \text{ ft by } y = \frac{32}{x^2} \]

\[ 4 \text{ ft by } 4 \text{ ft by } 2 \text{ ft} \]
Exercise: Find A and B such that \( y = Ax^{-1/2} + Bx^{1/2} \) has a minimum value of 6 at \( x=9 \).

\[ \begin{align*}
\text{min. value is 6:} & \quad f(9) = 6 \quad (9, 6) \\
\text{min. value is at } x=9 & \quad x=9 \text{ is a critical point} \\
\Rightarrow & \quad f'(9) = 0 \\
\text{2 pieces of information:} & \quad \begin{aligned}
& (1) \quad f(9) = 6 \\
& (2) \quad f'(9) = 0
\end{aligned}
\end{align*} \]

1. \( f(9) = 6 \Rightarrow A \cdot 9^{-1/2} + B \cdot 9^{1/2} = 6 \Rightarrow A \cdot \frac{1}{3} + B \cdot 9 = 6 \Rightarrow A + 9B = 18 \) multiply by 3

2. \( f'(9) = 0 \Rightarrow f'(x) = -\frac{1}{2} A \cdot x^{-3/2} + \frac{1}{2} B \cdot x^{-1/2} \Rightarrow f'(9) = -\frac{1}{2} A \cdot 9^{-3/2} + \frac{1}{2} B \cdot 9^{-1/2} = 0 \Rightarrow -\frac{1}{2} \cdot \frac{1}{9} A + \frac{1}{2} \cdot \frac{1}{3} B = 0 \Rightarrow -A + 9B = 0 \Rightarrow A = 9B \)
Put these two equations together (solve a system)

\[ A + 9B = 18 \]
\[ A = 9B \]

\[ 9B + 9B = 18 \]
\[ 18B = 18 \]
\[ B = 1 \]

\[ \Rightarrow A = 9, B = 9, 1 = 9 \]

\[ A = 9, \quad B = 1 \]