Math 1431
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Visit CASA regularly for announcements and course material!

If you e-mail me, please mention your course (1431) in the subject line.

Access code deadline:

Purchase popper scantrons from UH Bookstore.

Check online quiz due dates from CASA! No make ups.

Respect your friends! Do not distract anyone by chatting with people around you… Be considerate of others in class.

Note about I VT

If a function is not continuous on the given interval, then I VT does not apply. For example:

\[ f(x) = \frac{x-1}{x} \text{ over } [-1, 2] \]

Can’t use I VT for this function on this interval!
Section 1.5 – Intermediate Value Theorem

An important property of continuous functions is given in the following theorem.

**Theorem: The Intermediate Value Theorem**

If $f$ is a continuous function on the closed interval $[a,b]$, and $N$ is a real number such that $f(a) \leq N \leq f(b)$, then there is at least one number $c$ in the interval $(a,b)$ such that $f(c) = N$.

The diagram illustrates the theorem with a continuous function $y = f(x)$, a horizontal line at $y = N$, and points $(a, f(a))$ and $(b, f(b))$. The function value at $c$ is $f(c)$, which equals $N$. An example function $f(x) = 4x^5 - x^3 + x^2 - 2$ is given.
We can use the Intermediate Value theorem to prove the existence of roots (zeros or $x$–intercepts) of a function.

**Example:** Use the Intermediate Value theorem to show that the function has a root in the indicated interval.

\[ f(x) = x^3 - 6x^2 - x + 2, \quad [0, 1]. \]

1. \( f \) is continuous over \([0, 1]\). ✓
2. \( f(0) = 0 - 0 - 0 + 2 = 2 > 0 \) (+)
   \( f(1) = 1 - 6 - 1 + 2 = -4 < 0 \) (-)

Here, by IVT, \( f(x) \) has at least one root in \((0, 1)\).
Example: Show that there is a value between 1 and 2 so that \(-x^3 + 2x^4 = 7\).

\[
\begin{align*}
f(x) &= -x^3 + 2x^4 \\
f(1) &= -1^3 + 2 \cdot 1 = 1 \\
f(2) &= -8 + 2 \cdot 16 = 24
\end{align*}
\]

\[1 < 7 < 24\]

Therefore, by IVT, this equation has at least one solution in \((1, 2)\).

Note: \[f(x) = \frac{x-2}{x}, \quad [-1, 3]\]

IVT does not apply because \(f(x)\) is not continuous on this interval.
“Exercises” are left for you for practice. Please solve them after class. I may add the solutions to the notes later.

**Exercise:** Show that the following equation has a solution in the interval \( \left[ 0, \frac{\pi}{4} \right] \):

\[
2\tan(x) - x = 1.
\]

\[
f(x) = 2\tan(x) - x
\]

1. \( f \) is continuous on \( \left[ 0, \frac{\pi}{4} \right] \)

\[
f(0) = 0
\]

\[
f\left( \frac{\pi}{4} \right) = 2 \cdot 1 - \frac{\pi}{4} = 2 - \frac{\pi}{4} > 1
\]

\[
f(0) < 1 < f\left( \frac{\pi}{4} \right)
\]

By IVT, there is at least one \( c \) in \( \left( 0, \frac{\pi}{4} \right) \) such that \( f(c) = 1 \).
Another property of continuous functions is about extreme values.

**Theorem 1.5.2: The Extreme-Value Theorem**

If \( f \) is continuous on a bounded interval \([a,b]\), then \( f \) takes on both a maximum value and a minimum value.
Section 1.6 – The Pinching Theorem

Theorem: The Pinching Theorem

Let $p > 0$ and $c$ be a real number. Suppose that $f(x) \leq g(x) \leq h(x)$ for any $x$ in the interval $(c-p, c+p)$ (except possibly at $x = c$).

If $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$, then $\lim_{x \to c} g(x) = L$. 

\[c\]

\[c + p\]
Trigonometric Limits:

\[
\lim_{x \to 0} \sin x = \sin(0) = 0
\]

\[
\lim_{x \to 0} \cos x = \cos(0) = 1
\]

\[
\lim_{x \to \pi} (\cos x) = \cos(\pi) = -1
\]

\[
\lim_{x \to \frac{\pi}{2}} \left(\frac{\sin x}{4x}\right) = \frac{\sin(\frac{\pi}{2})}{4 \cdot \frac{\pi}{2}} = \frac{1}{2\pi}
\]

In general, \(\lim_{x \to c} \sin(x) = \sin(c)\) and \(\lim_{x \to c} \cos(c) = \cos(c)\).
Now, we look at two very important limits.

\[
\lim_{x \to 0} \frac{\sin(x)}{x} = ? \quad \text{and} \quad \lim_{x \to 0} \frac{1 - \cos(x)}{x} = ?
\]

If you try to use direct substitution on these limits, you’ll get the indeterminate form 0/0.

Idea: For the function \( g(x) = \frac{\sin x}{x} \), if \(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\), then \( \cos x \leq \frac{\sin x}{x} \leq 1 \).

Since \( \lim_{x \to 0} \cos x = 1 \) and \( \lim_{x \to 0} 1 = 1 \), by the Pinching theorem, we conclude that \( \lim_{x \to 0} \frac{\sin(x)}{x} = 1 \).
Fact: \( \lim_{x \to 0} \frac{\sin(x)}{x} = 1 \) and \( \lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0 \).

Fact: \( \lim_{x \to 0} \frac{\sin(ax)}{bx} = \frac{a}{b} \) and \( \lim_{x \to 0} \frac{1 - \cos(ax)}{bx} = 0 \).

ALWAYS pay attention to what value \( x \) is approaching.

Examples:

\[
\lim_{x \to 0} \frac{\sin(5x)}{x} = \frac{5}{1} = 5
\]

\[
\lim_{x \to 0} \frac{\sin(2x)}{6x} = \frac{2}{6} = \frac{1}{3}
\]

\[
\lim_{x \to 0} \frac{2x}{\sin(7x)} = \frac{2}{7}
\]
\[
\lim_{x \to 0} \frac{\sin(4x)}{\sin(9x)} =
\]

\[
\lim_{x \to 0} \frac{\sin^2(4x)}{5x} =
\]

\[
\lim_{x \to 0} \frac{\sin^2(4x)}{5x^2} =
\]

\[
\lim_{x \to \pi/2} \frac{\sin(2x)}{6x} =
\]
Examples:

$$\lim_{x \to 0} \frac{1 - \cos(2x)}{5x} =$$

$$\lim_{x \to 0} \frac{1 - \cos(4x)}{9x} =$$

$$\lim_{x \to \pi} \frac{1 - \cos(3x)}{6x} =$$