Math 1432 DAY 25

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Bubble in PS ID and Popper Number very carefully. If you make a bubbling mistake, your scantron can’t be saved in the system. In that case, you will not get credit for the popper even if you turned it in.

Check your CASA account for Quiz due dates. Don’t miss any online quizzes!

Be considerate of others in class. Respect your friends and do not distract anyone during the lecture.

POPPER#

Q# Evaluate: \[ \sum_{k=0}^{3} (k^3) \]

A) 0 B) 12 C) 27 D) 36 E) None

Q# Evaluate: \[ \sum_{k=1}^{4} (2^{k-1}) \]

A) 1 B) 15 C) 16 D) 33 E) None of these
Section 9.3 Infinite Series

Infinite Series

Let \( \{a_n\}_{n=0}^{\infty} \) be a sequence of real numbers.

**Finite Series:** \[ \sum_{k=0}^{n} a_k = a_0 + a_1 + a_2 + \ldots + a_n \]

**Infinite Series:** \[ \sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + \ldots + a_n + \ldots \]

**Examples:**

\[ \sum_{k=1}^{\infty} \frac{1}{k} \]

\[ \sum_{k=1}^{\infty} \frac{1}{k^2} \]

\[ \sum_{k=0}^{\infty} 2^{k+1} \]
Definition: Let
\[
\begin{align*}
\sum_{k=0}^{0} a_k &= a_0 \\
\sum_{k=0}^{1} a_k &= a_0 + a_1 \\
\sum_{k=0}^{2} a_k &= a_0 + a_1 + a_2 \\
\vdots \\
\sum_{k=0}^{n} a_k &= a_0 + a_1 + a_2 + \ldots + a_n \\
\end{align*}
\]

The corresponding sequence \(\{s_n\}\) is called the **Sequence of Partial Sums for the** series \(\sum_{k=0}^{\infty} a_k\).

**Definition 9.3.1**

If the sequence \(\{s_n\}\) of partial sums converges to a finite limit \(L\), we write
\[
\sum_{k=0}^{\infty} a_k = L
\]

and say that

the **series** \(\sum_{k=0}^{\infty} a_k\) **converges to** \(L\).

We call \(L\) the **sum** of the series. If the sequence of partial sums diverges, we say that

the **series** \(\sum_{k=0}^{\infty} a_k\) **diverges**.

**Remark**  It is important to note that the sum of a series is not a sum in the ordinary sense. It is a limit. \(\square\)
Example: Find the sequence of partial sums.

\[ \sum_{k=0}^{\infty} k = 0 + 1 + 2 + 3 + 4 + 5 + \ldots \]

\[ S_0 = 0 \]
\[ S_1 = 0 + 1 = 1 \]
\[ S_2 = 0 + 1 + 2 = 3 \]
\[ S_3 = 0 + 1 + 2 + 3 = 6 \]
\[ S_4 = 0 + 1 + 2 + 3 + 4 \]

General term: \[ S_n = 0 + 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \]

Sequence of partial sums: \[ S_n = \frac{n(n+1)}{2} \]

As \( n \to \infty \), \( S_n \to \infty \)

Seq. of p. sums is divergent.

\[ \Rightarrow \sum_{k=0}^{\infty} k \text{ is divergent.} \]
Example: Find the sequence of partial sums.

\[
\sum_{k=0}^{\infty} \left( \frac{1}{2} \right)^k = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots
\]

\[S_0 = 1\]

\[S_1 = 1 + \frac{1}{2}\]

\[S_2 = 1 + \frac{1}{2} + \frac{1}{4}\]

\[S_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 2 \left[ 1 - \left( \frac{1}{2} \right)^{n+1} \right] \]

\[S_n = 2 \left[ 1 - \left( \frac{1}{2} \right)^{n+1} \right] \]

sequence of partial sums.

\[n \to \infty \implies \left( \frac{1}{2} \right)^{n+1} \to 0\]

\[S_\infty \to 2 \left[ 1 - 0 \right] = 2\]

The series \(\sum_{k=0}^{\infty} \left( \frac{1}{2} \right)^k\) is Convergent

\[1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2\]
Definition: the series is convergent IF the sequence of partial sums converges to a real number. (not the generator sequence!!!)

<table>
<thead>
<tr>
<th>Generator Sequence:</th>
<th>Sequence of Partial Sums</th>
<th>Series</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_n = \frac{1}{2^n}$</td>
<td>$s_1 = 1$ $s_2 = 1 + \frac{1}{2}$ $s_2 = 1 + \frac{1}{2} + \frac{1}{4}$ $\vdots$ $s_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = \frac{1 - 2^{n+1}}{1 - \frac{1}{2}} = 2\left(1 - \left(\frac{1}{2}\right)^{n+1}\right)$</td>
<td>$\sum a_n = \sum \frac{1}{2^n}$ $= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$ $\rightarrow$ Easy to see that the sequence of partial sums converges to 2. The series converges to 2.</td>
</tr>
</tbody>
</table>

| Converges to 0 | Generator Sequence: $a_n = \frac{1}{2^n}$ | Sequence of Partial Sums $s_1 = 1$ $s_2 = 1 + \frac{1}{2}$ $s_3 = 1 + \frac{1}{2} + \frac{1}{4}$ $\vdots$ $s_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}$ $= 2^{n+1} - 1$ | Series $\sum a_n = \sum 2^n$ $= 1 + 2 + 4 + 8 + \cdots + 2^n + \cdots$ Divergent Easy to see that the sequence of partial sums is divergent. The series is divergent. |

$\sum \Rightarrow 2$ $\sum \Rightarrow \infty$
| Generator Sequence: $a_n = \frac{1}{n}$ | Sequence of Partial Sums  
$s_1 = 1$

$s_2 = 1 + \frac{1}{2}$

$s_3 = 1 + \frac{1}{2} + \frac{1}{3}$

\[ \vdots \]

$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$ | Series  
$\sum a_n = \sum \frac{1}{n}$

$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} + \ldots$ |

| Converges to 0 | Not easy to see whether sequence of partial sums is convergent or not.  

????? | |

**Definition:** The series is convergent IF the sequence of partial sums converges to a real number. (not the generator sequence!!!)

But it is hard to tell whether the sequence of partial sums converges or not. That’s why we don’t use this definition much. We have tests to decide on convergence of a series.

Given a series $\sum_{k=0}^{\infty} a_k$, how do we know whether the series converges to a finite number or diverges?

Write down the sequence of partial sums; check if this sequence converges or not. Sometimes, this is long and/or difficult.
\( a_k = k^2 \)

\[ \{1, 4, 9, 16, 25, \ldots \} \]

\( \sum_{k=1}^{3} k^2 = 1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14 \)

\( \sum_{k=1}^{\infty} k^2 = 1 + 4 + 9 + 16 + 25 + \ldots \)

\( \sum_{k=1}^{\infty} \left( \frac{1}{k^2} \right) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots \)
First, a theorem:

**Theorem**: If \( \sum_{k=0}^{\infty} a_k \) converges, then \( a_k \to 0 \) as \( k \to \infty \).

If the series is convergent, then the generator sequence must converge to 0.

Conversely: if the generator sequence does not converge to 0, then the series cannot be convergent.

As a result of this theorem, we get a very useful test:

**BASIC DIVERGENCE TEST (BDT)**–

If \( \{a_n\}_{n=0}^{\infty} \) is a sequence such that \( \lim_{n \to \infty} \{a_n\} \neq 0 \), then the series \( \sum_{k=0}^{\infty} a_k \) DIVERGES.

Note: This test can only be used to prove divergence. It does not prove convergence of a series.
Example: Divergent or convergent?

\[
\sum_{k=0}^{\infty} 5k 
\]
Divergent by BDT \( (5k \not\to 0) \)

\[
\sum_{k=0}^{\infty} \frac{k + 5k^2}{k^2 + 2} 
\]
Divergent by BDT \( \left( \frac{k + 5k^2}{k^2 + 2} \not\to 0 \right) \)

\[
\sum_{k=0}^{\infty} \frac{1}{k} 
\]
BDT doesn't apply.

\[
\sum_{k=0}^{\infty} \frac{1}{2^k} 
\]
BDT doesn't apply.

\[
\frac{1}{2^k} \to 0
\]
Geometric Series

A series of the form \[ \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \ldots \]

where \( x \) is a real number is called a geometric series.

Examples: \[ \sum_{k=0}^{\infty} 2^k \] is a geometric sequence where \( x=2 \).

\[ \sum_{k=0}^{\infty} \left( \frac{1}{5} \right)^{k+1} \] is a geometric sequence where \( x=1/5 \).

\[ \sum_{k=0}^{\infty} \frac{1}{k} \text{ or } \sum_{k=0}^{\infty} k \cdot 2^k \] are NOT geometric series.

The general term for the sequence of partial sums is:

\[ s_n = \sum_{k=0}^{n} x^k = 1 + x + x^2 + x^3 + \ldots + x^n = \frac{1-x^{n+1}}{1-x} \]

For the series \[ \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \ldots \], the sum of the first \( n \) terms is:

\[ s_n = \frac{(1-x^{n+1})}{1-x} \]

If \( |x| < 1 \), then \( x^{n+1} \to 0 \) as \( n \to \infty \). That is, \[ \lim_{n \to \infty} \frac{(1-x^{n+1})}{1-x} = \frac{1-0}{1-x} = \frac{1}{1-x} \].
FACT:

If $|x| < 1$, then the series $\sum_{k=0}^{\infty} x^k$ is convergent; $\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}$.

NOTE: If the first term is not 1; then $\sum_{k}^{\infty} x^k = \frac{\text{first term}}{1 - x}$.

If $|x| \geq 1$, then the series $\sum_{k=0}^{\infty} x^k$ is divergent.

Examples: Determine if the series are convergent or not.

$\sum_{k=0}^{\infty} 5^k$ Divergent, geo series with $x = 5 > 1$

$\sum_{k=0}^{\infty} \left(\frac{2}{5}\right)^k$ Convergent, geo series with $x = \frac{2}{5} < 1$

$\sum_{k=0}^{\infty} \left(\frac{2}{5}\right)^k = 1 + \frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \cdots$

$= \frac{1}{1 - \frac{2}{5}} = \frac{5}{3} = \sqrt{\frac{5}{3}}$
Example: Determine if the series are convergent or not. If possible, find the sum:

\[
\sum_{k=1}^{\infty} \frac{2^n}{3^{n+1}} = \sum_{n=1}^{\infty} \left( \frac{2}{3} \right)^n \cdot \frac{1}{3} = \left( \frac{2}{9} \right) + \frac{4}{27} + \ldots
\]

Conv.

\( x = \frac{2}{3} < 1 \)

Example: Determine if the series are convergent or not. If possible, find the sum:

\[
\sum_{k=1}^{\infty} \frac{3^k}{5^{k+1}}
\]

\[
\sum \frac{9^n}{5^{n+1}} = \sum \left( \frac{9}{5} \right)^n \cdot \frac{1}{5}
\]

\( x = \frac{9}{5} > 1 \)

Geo. divergent
Popper #

Choose (C) for convergent  (D) for divergent.

Q# \[ \sum_{k=1}^{\infty} \left( \frac{5}{4} \right)^k \]

Q# \[ \sum_{k=1}^{\infty} \left( \frac{-1}{4} \right)^{k+2} \]

Q# If possible, find the sum of the series: \[ \sum_{k=1}^{\infty} \left( \frac{1}{4} \right)^k \]

A) \( \frac{1}{3} \)  B) \( \frac{4}{3} \)  C) \( \frac{3}{4} \)  D) Divergent  E) none of these