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Bubble in PS ID and Popper Number very carefully. If you make a bubbling mistake, your scantron can’t be saved in the system. In that case, you will not get credit for the popper even if you turned it in.

Check your CASA account for Quiz due dates. Don’t miss any online quizzes!

Be considerate of others in class. Respect your friends and do not distract anyone during the lecture.

Popper#

Q# Give the 7th degree Taylor polynomial approximation for  
\[ f(x) = \sin(x) \] centered at \( x = 0 \).

a. \[ x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} \]

b. \[ 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} \]

c. \[ 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \]

d. \[ x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \]
Q# Give the 7th degree Taylor polynomial approximation for $f(x) = \cos(x)$ centered at $x = 0$.

a. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7}$

b. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!}$

c. $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$

d. $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

Q# Give the 7th degree Taylor polynomial approximation for $f(x) = \ln(x+1)$ centered at $x = 0$.

a. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7}$

b. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!}$

c. $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$

d. $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$
Section 9.8 Taylor Series

Know these:

\( n^{\text{th}} \) degree Taylor Polynomials centered at \( x=0 \):

\[ e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} \quad \text{for all } x. \]

\[ \cos x \approx 1 \ - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} \quad \text{for all } x. \]

\[ \sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{for all } x. \]

\[ \ln(1 + x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n+1} \frac{x^n}{n} \quad \text{for } -1 < x \leq 1. \]
Lagrange Form of the Remainder

or

Lagrange Error Bound or Taylor’s Theorem Remainder

When a Taylor polynomial is used to approximate a function, we need a way to see how accurately the polynomial approximates the function.

\[ f(x) = P_n(x) + R_n(x) \quad \text{so} \quad R_n(x) = f(x) - P_n(x) \]

Written in words:

Function = Polynomial + Remainder

\[ f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \ldots \]

so

Remainder = Function – Polynomial

Lagrange Formula for Remainder:

Suppose \( f \) has \( n+1 \) continuous derivatives on an open interval that contains 0. Let \( x \) be in that interval and let \( P_n(x) \) be the \( n \)th Taylor Polynomial for \( f \). Then

\[ R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \]

where \( c \) is some number between 0 and \( x \).
If we rewrite Taylor’s theorem using the Lagrange formula for the remainder, we have

\[
f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}
\]

where \(c\) is some number between 0 and \(x\).

If there is a number \(M\) so that

\[
|f^{(n+1)}(c)| \leq M
\]

for all \(c\) between 0 and \(x\) then

\[
|f(x) - P_n(x)| \leq \frac{M}{(n+1)!}x^{n+1}
\]

or

\[
|R_n(x)| \leq \left( \max \left| f^{(n+1)}(c) \right| \right) \frac{x^{n+1}}{(n+1)!}
\]

for some \(c\) between 0 and \(x\).

We probably will not know the value of \(c\).
Example: Give an error estimate for the approximation of \( \sin(x) \) by 9th degree Taylor polynomial \( P_9(x) \) for an arbitrary value of \( x \) between 0 and \( \pi/4 \), centered at \( x = 0 \).

\[
|R_n(x)| \leq \left( \max |f^{(n+1)}(c)| \right) \frac{|x|^{n+1}}{(n+1)!}
\]

\[
f(x) = \sin x \\
f''(x) = \cos x \\
f'''(x) = -\sin x \\
f^{(4)}(x) = -\cos x \\
f^{(5)}(x) = \sin x
\]

Example: Assume that \( f \) is a function such that \( |f^{(n)}(x)| \leq 1 \) for any \( n \) and \( x \). Estimate the error if 7th degree Taylor polynomial \( P_7(-2) \) is used to approximate \( f(-2) \).

Example: Give the Lagrange form of the remainder:

\[
f(x) = \sin(2x) \quad , \quad n = 3.
\]
TAYLOR SERIES in $x$

If $f$ is infinitely differentiable on an interval $I$ containing 0, then we say that “$f$ can be expanded as a Taylor series in $x$” and write:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

Taylor Series for some basic functions:

- $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ for all real $x$.

- $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ for all real $x$.

- $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ for all real $x$.

- $\ln(1 + x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ for $-1 < x \leq 1$. 
Example: We know: \[ f(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \]

Find the Taylor series for \( g(x) = e^{x^2+1} \) centered at 0.

Find the Taylor series for \( g(x) = xe^{2x} \) centered at 0.
**Example:** We know: \( f(x) = \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \)

Find the Taylor series for \( g(x) = \sin(2x) \) centered at 0.

Find the Taylor series for \( h(x) = x \sin(x^2) \) centered at 0.

**Remark:** “Expand \( f(x) = e^x \) in powers of \( (x-1) \)” means: **write the power (Taylor) series for this function with center \( x=1 \).**

**Exercise:** Expand \( f(x) = e^{2x} \) in powers of \( (x-1) \).