completed

**Math 2413 - Review for Test 2**

Schedule your exam; students are responsible for scheduling their tests BEFORE the testing window begins. Do not be late for your test. If you're more than 10 minutes late, CASA will not let you in. Your instructor does not control the scheduler; don't miss your scheduled time. If you miss it, try rescheduling online; you can do this only if there are available seats.

Material: Chapter 3

When: By reservation; check your confirmation email.

Where: CASA Testing center (check your confirmation email for exact location)

What to bring: Picture ID, writing utensils.

Time: 75 minutes

Number of questions: Approximately 18.

MC: TBA

FR: 3-4 questions

The grade you'll see right away will be for MC part only (out of ?? points).

Take practice test 2! 5% of your best score will be added to your test grade.

**How to study?**

Go over class notes, rework past HW assignments and labquizzes. Work on the main review sheet posted on CASA and take the practice test. Solve all problems on this review sheet. Check the keys for WHW (posted on CASA) to see how to show work properly.

What to know:

- Related Rates
- How to determine the intervals of increase/decrease, concave up/down.
- Critical points, local min/max, absolute min/max, points of inflection.
- First derivative test, Second derivative test.
- Given the graph of f' , what can you conclude about f ?
- Given the graph of f'', what can you conclude about f ?
- Finding vertical tangents, cusps.
- Graphing a function (with all details).

**Related Rates**

- Work all the problems in your homework and quizzes (solve: shadow length, ladder, area of a rectangle, area of a triangle, area of a circle, volume of a sphere, volume of a cone, etc….)
- Must know basic area and volume formulas from geometry.
  Area: Circle, Square, Triangle, Rectangle
  Surface area: Cube, Rectangular Prism
  Volume: Cube, Cone, Cylinder, Sphere, Rectangular Prism

$$2 \cdot h \cdot x^3 \leftarrow \qquad 2 \cdot h \cdot x^3$$

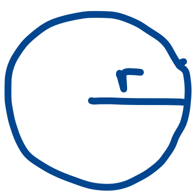
1. Given $K = 2hx^3$. Find the derivative of $K$ with respect to time $t$; assume that h stays constant with time, and $x$ varies with time.

$$\frac{dK}{dt} = 2h \cdot 3 \cdot x^2 \cdot \frac{dx}{dt} = 6hx^2 \frac{dx}{dt}$$

what if h changes but x stays constant

$$\frac{dK}{dt} = 2 \cdot \frac{dh}{dt} \cdot x^3 = 2x^3 \cdot \frac{dh}{dt}$$

2. The radius of a circle is increasing at a rate of 2inches/sec. How fast is the perimeter changing when the radius is 10 inches?




Given: $\frac{dr}{dt} = 2$ in/sec

$$P = 2 \cdot \pi \cdot r$$

deriv. of both sides (dt)

$$\frac{dP}{dt} = 2\pi \cdot \frac{dr}{dt}$$

given

$$\left.\frac{dP}{dt}\right|_{r=10} = 2\pi \cdot 2 = \boxed{4\pi} \text{ in/sec.}$$

what if

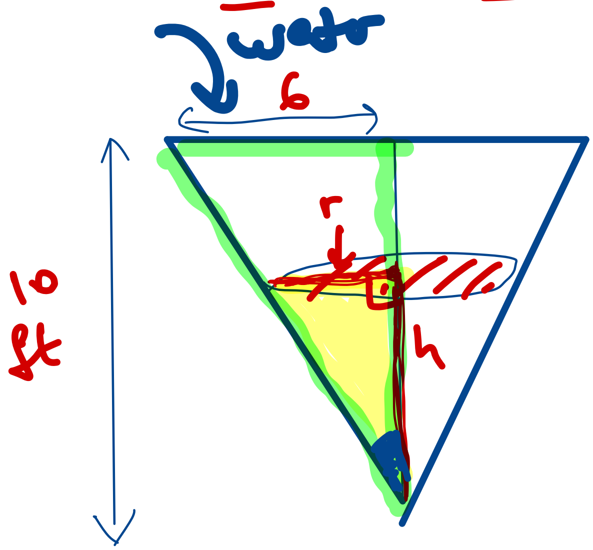
$$A = \pi \cdot r^2$$

deriv.

$$\frac{dA}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt}$$

**Solve all related rates questions on the main review sheet.**

**3.** Water is poured into a conical tank at the rate of 64 cubic feet per minute. If the tank is 10 feet tall and the top of the tank has a radius of 6 feet, how fast does the radius of the water surface change when the water is 2 feet deep?




Given: $\frac{dV}{dt} = 64 \text{ ft}^3/\text{min}$

Question: $\left.\frac{dr}{dt}\right|_{h=2 \text{ ft}} = ?$

depth of the water

$V_{cone} = \frac{1}{3} \cdot \pi \cdot r^2 \cdot h$

$\frac{6}{10} = \frac{r}{h} \Rightarrow 6h = 10 \cdot r$

$\boxed{3h = 5r}$

$\boxed{h = \frac{5r}{3}}$

substitute: $V = \frac{1}{3} \cdot \pi \cdot r^2 \cdot \left(\frac{5r}{3}\right)$

rewrite: $V = \frac{5}{9} \pi \cdot r^3$

Take derivatives of both sides wrt time:

$$\frac{dV}{dt} = \frac{5}{\cancel{9}_{3}} \cdot \pi \cdot \cancel{3} \cdot r^2 \cdot \boxed{\frac{dr}{dt}}$$

$$\frac{dV}{dt} = \underbrace{\frac{5\pi}{3} \cdot r^2}_{} \cdot \boxed{\frac{dr}{dt}}$$

?

$$\frac{dr}{dt} = \frac{3}{5\pi r^2} \cdot \frac{dV}{dt}$$

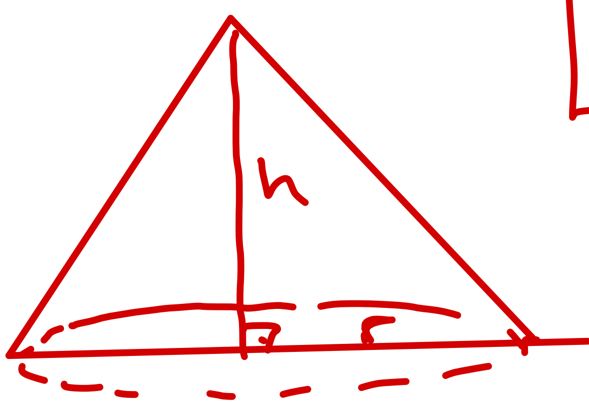
given

when $\underline{\underline{h=2}}$, $r = \frac{6}{5}$

$$\left.\frac{dr}{dt}\right|_{h=2} = \frac{3}{5 \cdot \pi \cdot \left(\frac{6}{5}\right)^2} \cdot 64$$

$$\left.\frac{dr}{dt}\right|_{h=2} = \frac{\cancel{3} \cdot \cancel{6}4}{\cancel{36}_{12} \cdot \pi / 5} = \boxed{\frac{16 \cdot 5}{3\pi}} = \frac{80}{3\pi}$$

4. Sand is falling onto a conical pile at a rate of 40 cubic feet per minute. If the diameter of the base of the cone is 4 times its altitude. At what rate is the height of the pile changing when it is 20 feet high?




$\boxed{2 \cdot r = 4 \cdot h} \Rightarrow \boxed{r = 2h}$

quest. $\underbrace{\frac{dh}{dt}}\Big|_{h=20} = ?$

given: $\underbrace{\frac{dV}{dt}} = 40 \; ft^3/min$

$V = \frac{1}{3}\pi \cdot r^2 \cdot h$

substitute $r = 2h$

$V = \frac{1}{3}\pi \cdot (2h)^2 \cdot h = \frac{4}{3}\pi \cdot h^3$

$V = \frac{4}{3}\pi \cdot h^3$

$\downarrow \qquad \downarrow$ Take deriv. of both sides

$$\frac{dV}{dt} = \frac{4}{\cancel{3}} \cdot \pi \cdot \cancel{3} \cdot h^2 \cdot \frac{dh}{dt}$$

$$\frac{dV}{dt} = 4\pi \cdot h^2 \cdot \boxed{\frac{dh}{dt}}$$

$$\Rightarrow \frac{dh}{dt} = \frac{1}{4\pi \cdot h^2} \cdot \frac{dr}{dt}$$

$$\left.\frac{dh}{dt}\right|_{h=20} = \boxed{\frac{1}{4 \cdot \pi \cdot (20)^2} \cdot 40}$$

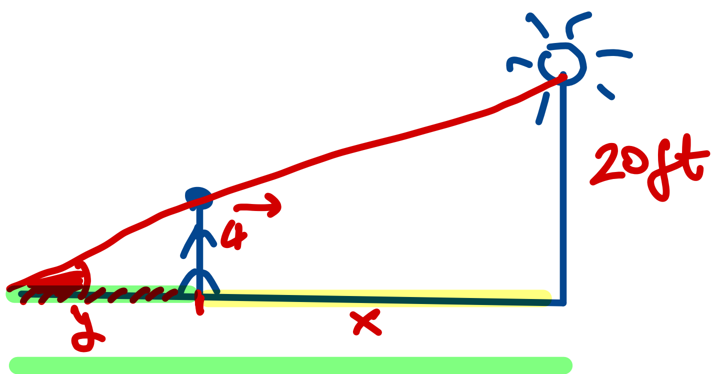
$$= \frac{\overset{2}{\cancel{40}}}{4 \cdot \pi \cdot \cancel{20} \cdot \underset{10}{\cancel{20}}}$$

$$= \boxed{\frac{1}{40\pi}}$$ ← answe

x: decreasing

5. Homer is 4 ft tall and he is walking towards a lamp post that is 20 ft high; if he is approaching the lamp at a rate of ½ feet/sec, how fast is his shadow changing when he is 5 feet away from the lamp?

decr.




Given: $\frac{dx}{dt} = -\frac{1}{2}$ ft/sec

Q: $\left.\frac{dy}{dt}\right|_{x=5} = ?$

$$\frac{4}{y} = \frac{20}{x+y} \Rightarrow 4x + 4y = 20y$$

$$\Rightarrow 4x = 16y$$

$$\Rightarrow \boxed{x = 4y}$$

⇒ take deriv. of both sides:

$$\boxed{\frac{dx}{dt}} = 4 \cdot \boxed{\frac{dy}{dt}}$$

$$\Rightarrow \frac{dy}{dt} = \frac{1}{4} \cdot \frac{dx}{dt} = \frac{1}{4} \cdot \frac{-1}{2}$$

given

$$= \boxed{\frac{-1}{8} \text{ ft/sec}}$$

Exercise: A 5 ft ladder is leaning against a wall. If it is pulled away from the wall from the bottom tip at a rate of 0.5 ft/sec, how fast is the height of the ladder on the wall changing when the bottom tip is 4 feet away from the wall?

Exercise: The altitude of a triangle is increasing at a rate of 5 in/min while the base is decreasing at a rate of 1 in/min. At what rate is the area of the triangle changing when the altitude is 10 in and the base is 6 in?

S3.1

6. A particle is moving on the x-axis with position function $x(t) = -t^3 + t^2 + 1$.

What is the **velocity** of this particle when t=5?

always $\geq 0$

What is the **speed** of the particle when t=5?

What is the **acceleration** of the particle when t=5?

$v(t) = x'(t) = \boxed{-3t^2 + 2t}$

$t=5$: $v(5) = -3(5)^2 + 2\cdot 5 = -65$

speed when $t=5$: $|-65| = \boxed{65}$

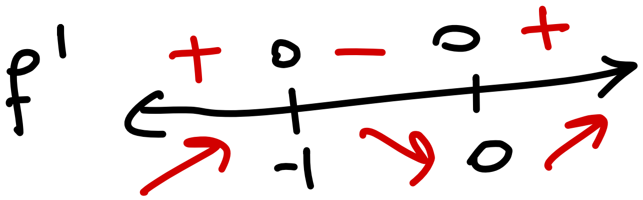
$a(t) = v'(t) = -6t+2$ $\qquad a(5) = -6\cdot 5 + 2 = -28$

7. Given: $f(x)=2x^3+3x^2-4$. Find the intervals over which this function is increasing.

$f'(x) = 6x^2 + 6x = 0$

$6x(x+1) = 0$

cp: $x = 0$, $x = -1$



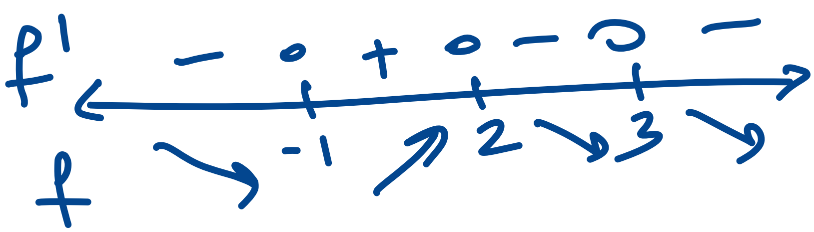

increasing

$(-\infty, -1)$ & $(0, \infty)$

8. Given: f(x) is a polynomial with the **first derivative**: $f'(x)=-2(x+1)(x-2)(x-3)^2$. Find the intervals over which this function is decreasing.

f(x)

cp: $-1$, $2$, $3$




decreasing on: $(-\infty, -1)$ & $(2, \infty)$

9. Given: $f(x)=5x^4+5x^3-4x+1$. Find the intervals over which this function is concave up.

$$f'(x)=20x^3+15x^2-4$$

$$f''(x)=60x^2+20x=30x(2x+1)=0$$

$x=0,\quad x=\frac{-1}{2}$

$f''$: $+$ 0 $-$ 0 $+$

$\cup$ $-\frac{1}{2}$ $\cap$ $0$ $\cup$

$\cup$: $(-\infty,-\frac{1}{2})$ & $(0,\infty)$

10. Given f(x) is a polynomial with **second derivative function**: $f''(x)=5(x+2)(x+1)(x-4)$. Find the intervals over which this function is concave down.

$-2,\ -1,\ 4$

$f''$: $-$ 0 $+$ 0 $-$ 0 $+$

$f$: $\cap$ $-2$ $\cup$ $-1$ $\cap$ $4$ $\cup$

$\cap$: $(-\infty,-2)$ & $(-1,4)$

11.Given: $f(x) = x^3 + 6x^2 - 9$.

Find the critical numbers and classify the them.

$$f'(x) = 3x^2 + 12x = 0$$

$$3x(x+4) = 0$$

cp: $x = 0, \quad x = -4$

f'   +   0   −   0   +

$\boxed{-4}$ local max     $\boxed{0}$ local min.

Radical $(-\infty, \infty) \rightarrow$

12. Find the critical numbers; Classify the critical points.

$$f(x) = \left(x^2 - 6x\right)^{1/3}$$

$$f'(x) = \frac{1}{3}(x^2 - 6x)^{-2/3}(2x - 6)$$

$$f'(x) = \frac{2x - 6}{3(x^2 - 6x)^{2/3}} = \frac{2(x-3)}{3(x(x-6))^{2/3}}$$

$f' = 0$ if $\boxed{x = 3}$

$f'$: undefined if $\boxed{x = 0, \quad x = 6}$

cp: $x = 0, 3, 6$.

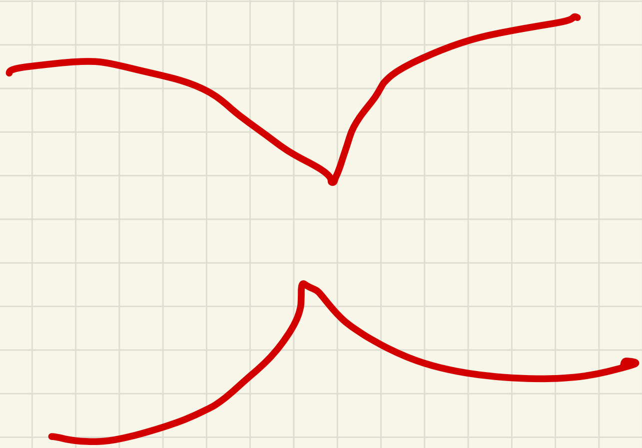
$f'$   $-$   undef   $-$   $0$   $+$   undef   $+$

$\searrow$ $\boxed{0}$ $\searrow$ $\boxed{3}$ $\nearrow$ $\boxed{6}$ $\nearrow$

$0$: no sign change, vertical tangent

$3$: local min

$6$: Vert. tangent



$f'$ − undef +

12

radical

13. Given: $f(x) = (x-12)^{2/3}$; classify the point $(12, f(12))$.

a) Local max
b) Point of inflection
c) Vertical tangent
d) Vertical cusp
e) None of the above.

$f'(x) = \frac{2}{3}(x-12)^{-1/3}$

$f'(x) = \dfrac{2}{3(x-12)^{1/3}}$

$x = 12$ is a c.p.

$\sqrt[3]{x-12}$

14. Given $f(x) = \frac{1}{20}x^5 - \frac{1}{4}x^4 + 2$, State the x-coordinates of any points of inflection.

$f'(x) = \frac{1}{4}x^4 - x^3$

$f''(x) = x^3 - 3x^2 = x^2(x-3)$

$f''$: − 0 − 0 +

$f$: ∩ 0 ∩ 3 ∪

Not a PoI

PoI at $\boxed{x=3}$

15. Find absolute minimum and maximum values (if any) of $f$ on the given interval and state where those values occur.

$f(x) = 2x^3 - 3x^2 + 2 \qquad [-1,2]$

$f'(x) = 6x^2 - 6x = 0$

$6x(x-1) = 0$

cp: $x=0$, $x=1$

in $[-1,2]$?

$f'$: $+ \; 0 \; - \; 0 \; +$ at $0$ and $1$

local max at $x=0$

local min at $x=1$

absolute

table of values for comparison

| $x$ | $f(x) = 2x^3 - 3x^2 + 2$ | |
|---|---|---|
| $-1$ | $-2-3+2 = \boxed{-3}$ | ← abs min |
| $0$ | $\boxed{2}$ | ← local max |
| $1$ | $2-3+2 = \boxed{1}$ | ← local min |
| $2$ | $16-12+2 = \boxed{6}$ | ← abs max |

abs min value is $-3$ at $(-1,-3)$

abs max value is $6$. at $(2,6)$.

second deriv. test

16. Given $f(x)$ is a polynomial with:

$f(2)=4$, $\boxed{f'(2)=0}$, $f''(2)=-5$. Classify the point $(2,4)$.

$\cup$ on R.

x=2 is a critical pt.

a) Intercept
b) Local min
c) Local max
d) Point of inflection
e) Vertical cusp
f) None of the above.

$f'' < 0$

2nd deriv test → if $f''(c) > 0$ then c is local min.

17. If the function $f(x)=Bx^3+4x+1$ has a critical point at $x=2$, find the value of B.

$f'(2)=0$

given

$f'(x) = 3.B.x^2 + 4$

$f'(2) = 3.B.2^2 + 4 = 0$

$12B + 4 = 0 \Rightarrow 12B = -4$

$B = \frac{-4}{12}$

$\boxed{B = -\frac{1}{3}}$

$$\frac{f(b)-f(a)}{b-a}$$

(*)

=

18. Verify that Mean Value Theorem applies. Find the c satisfying the Mean Value theorem on this interval.

$f(x)=x^3-x+1$ over [1,3].

MVT: There is at least one c such that

$$f'(c) = \frac{f(3)-f(1)}{3-1} = \frac{25-1}{2} = 12$$

Solve: $f'(x) = 12$

$3x^2-1=12 \Rightarrow 3x^2=13$

$\Rightarrow x^2 = \frac{13}{3}$

$\Rightarrow x = \pm\sqrt{\frac{13}{3}}$

in (1,3) ; $\boxed{c=\sqrt{\frac{13}{3}}}$

Exercise: Does MVT apply over the given interval?

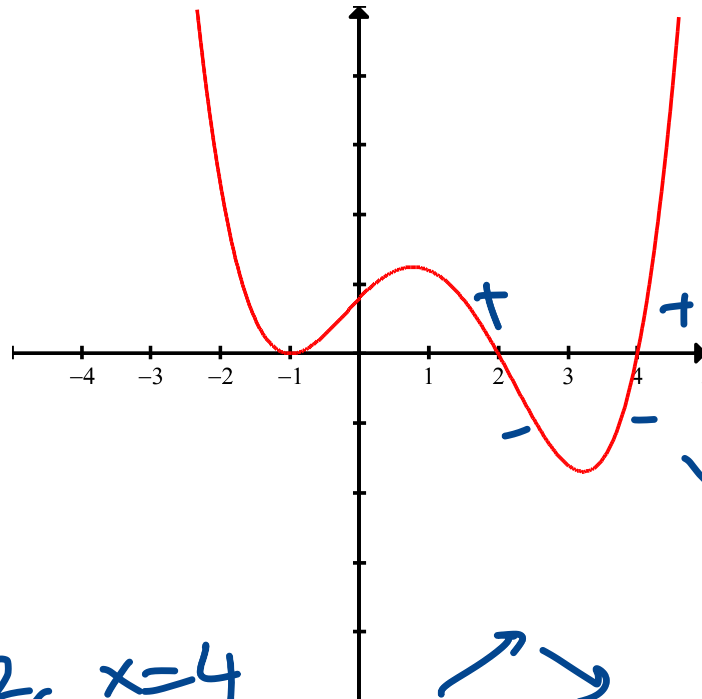
$f(x)=x^3+x$ over [1,3].

$f(x)=\sqrt{x-2}$ over [1,3].

$f(x)=\dfrac{x}{x-2}$ over [1,3].

$f(x)=2|x-1|$ over [0,3].

19. The graph of $f'(x)$ is given.




When is f(x) increasing?

$f' > 0$

$(-\infty, 2)$ & $(4, \infty)$

When is f(x) decreasing?

$f' < 0$

$(2, 4)$

Critical numbers of f(x):

$x = -1$, $x = 2$, $x = 4$

Local min/max points for f(x):

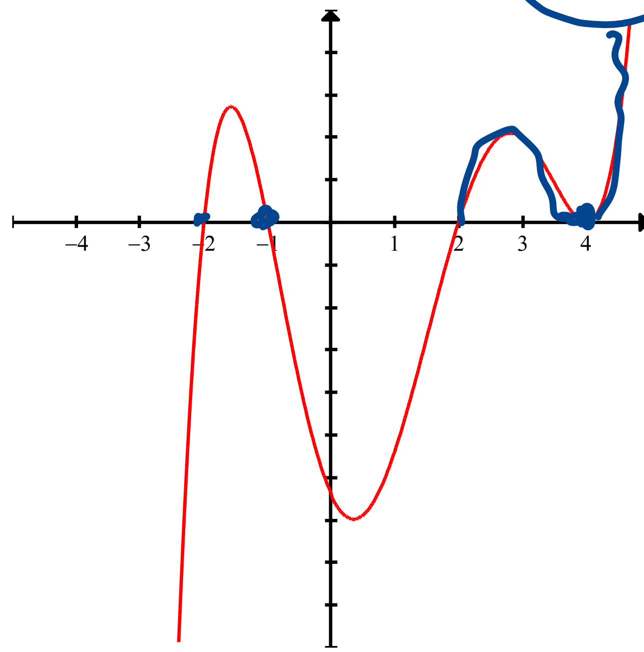
$x = -1$ Neither

$x = 2$ local max

$x = 4$ local min.

$(2, 4) \;\&\; (4, \infty)$

20. The graph of $f''(x)$ is given.




When is f(x) concave up?

$f'' > 0$

$(-2, -1) \;\&\; (2, \infty)$

When is f(x) concave down?

Points of inflection for f(x):

$x = -2, \quad x = -1, \quad x = 2, \quad \cancel{x = 4}$

$\underbrace{x=-2,\ x=-1,\ x=2}_{\text{PoI}}$

$f \cup$ if $f' \nearrow$

$f \cap$ if $f' \searrow$

21. State the domain, any asymptotes, critical numbers, the intervals where the function is increasing/decreasing, concave up/down, and any Points of inflection.

$$\boxed{f(x)=\frac{-10x}{(x-4)^2}}$$

$$\frac{-10x}{x^2}\ldots$$

Derivatives:

$$f'(x)=\frac{10x+40}{(x-4)^3} = \frac{10(x+4)}{(x-4)^3}$$

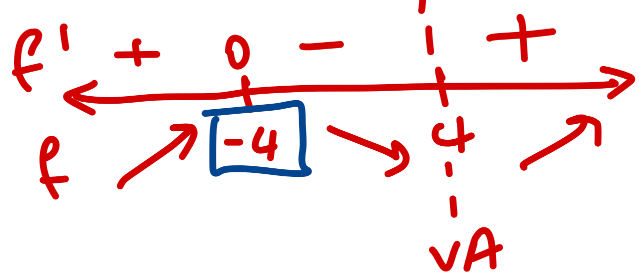
$$f''(x)=\frac{-20x-160}{(x-4)^4}$$

Domain: $(-\infty,4)\cup(4,\infty)$

Asymptotes: VA: $x=4$ ; HA: $y=0$

Intercepts: $x=0$ $(0,0)$

$f'=0 \therefore x=-4$




Critical Points: $\boxed{x=-4}$

The function is Increasing on:

The function is Decreasing on:

Local extreme points:

Increasing on: $(-\infty,-4)$ & $(4,\infty)$

Decr. on: $(-4,4)$

local max at $(-4, f(-4))$

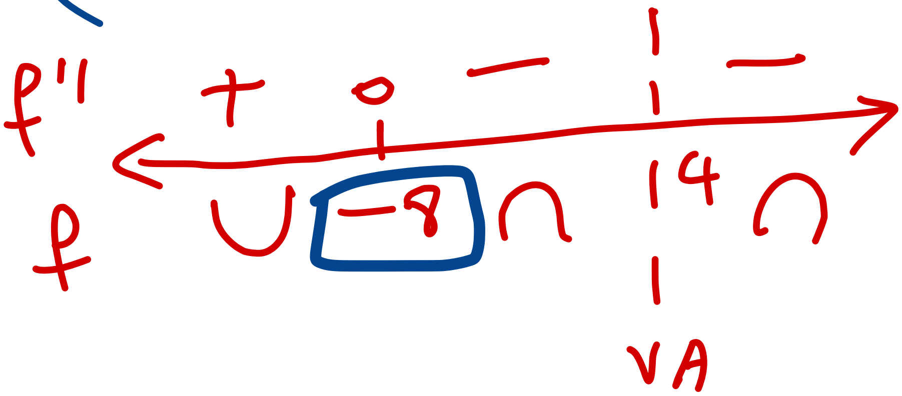
$$\frac{40}{64}=\frac{5}{8}$$

$$f''(x) = \frac{-20(x+8)}{(x-4)^4}$$

The function is Concave up on:

The function is Concave down on:

Points of inflection: ___________________

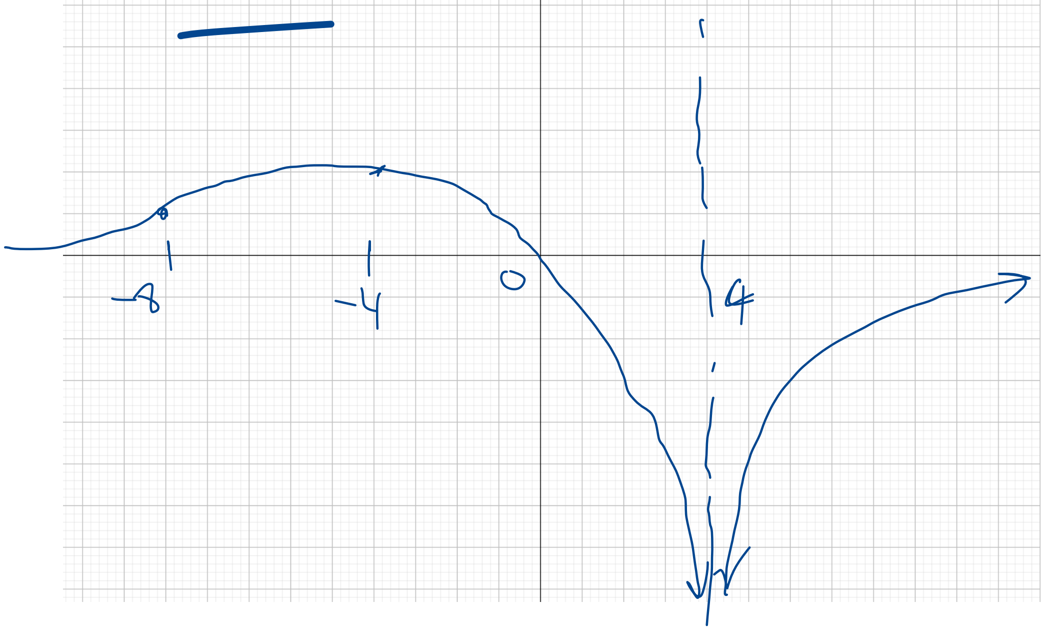



Concave up on: $(-\infty, -8)$

Exercise: Graph the function.

Concave down on: $(-8, 4)$ & $(4, \infty)$

PoI at $x = -8$ $\quad (-8, f(-8))$



EXERCISE: State the domain, any asymptotes, critical numbers, the intervals where the function is increasing/decreasing, concave up/down, and GRAPH this function.

$$f(x) = \frac{x^2}{(x+2)^2}$$

$$f'(x) = \frac{4x}{(x+2)^3}$$

$$f''(x) = \frac{-8x+8}{(x+2)^4}$$

EXERCISE:

What can you say about a function with these properties:

1. The domain is all real numbers except 3 and -3.
2. The function has vertical asymptotes at $x = 3$ and $x = -3$
3. The function is symmetric about the y-axis
4. $\lim_{x \to \infty} f(x) = -1$
5. $f(0) = 0, f(2) = 0, f(4) = 0$
6. $f'(x) < 0$ for $0 < x < 1$ and $x > 3$
7. $f'(x) > 0$ for $1 < x < 3$
8. $f''(x) < 0$ for $0 < x < 1/2$

**POPPER#**

**Question#** Is there a vertical tangent or a vertical cusp at the critical number for $f(x)=(x+3)^{\frac{2}{3}}$?

a. Vertical tangent b. cusp

$f'$ − undef +

−3

**Question#** Is there a vertical tangent or a vertical cusp at the critical number for $f(x)=(x-5)^{\frac{1}{5}}$?

a. Vertical tangent b. cusp

$f'$ + undef +

5

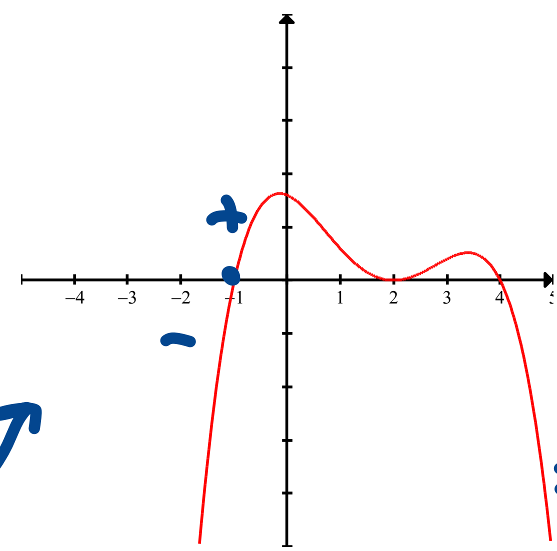
**Question#** Classify x = -1 if the graph is the derivative of *f*.

a. relative maximum

b. relative minimum

c. POI

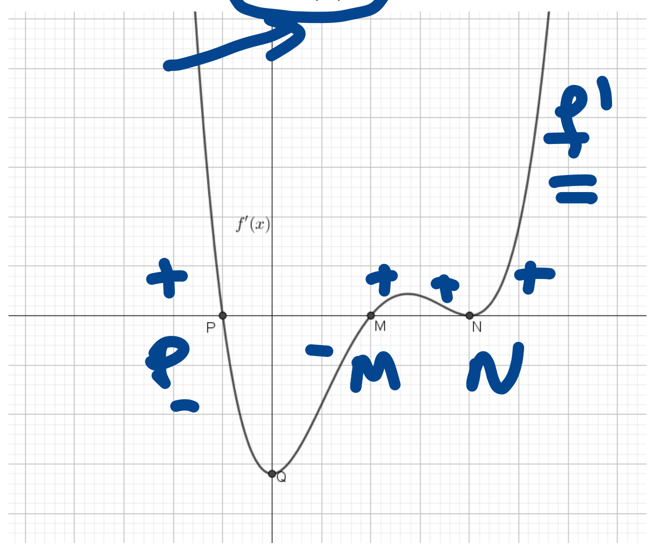
d. who cares?




$f'$

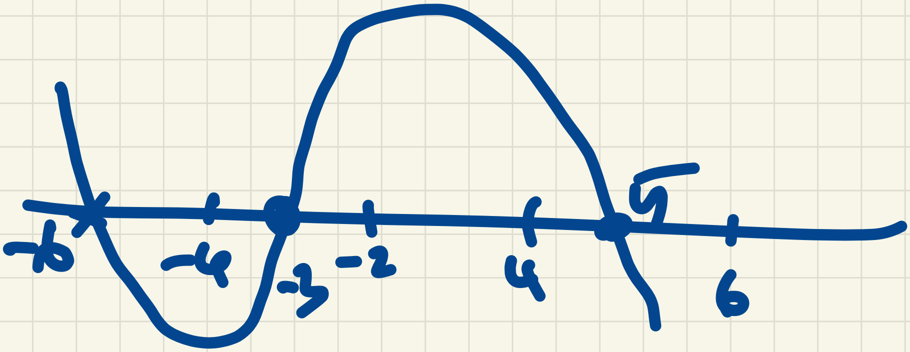
Question#  The graph of **the first derivative of** f(x) is given below (note the derivative is given!). Which point is a local minimum for f(x)?

a) P
b) M
c) N
d) Q
e) None of the above

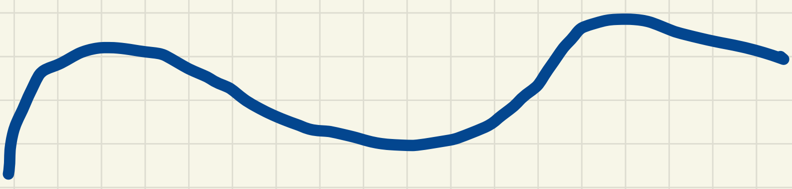



Take the practice test!

Good luck!




f ↗ ↘ ↗ ↘



$$f'(x) = \square\,(x-1)(x-2)(x-3)^2$$

$f'$: $+$ $0$ $-$ $0$ $+$ $0$ $+$

$1 \quad 2 \quad 3$

$$f'(x) = -5\,(x-1)(x-4)^2$$

$+$ $0$ $-$ $0$ $-$

$1 \quad 4$