Math 2413- Calculus I

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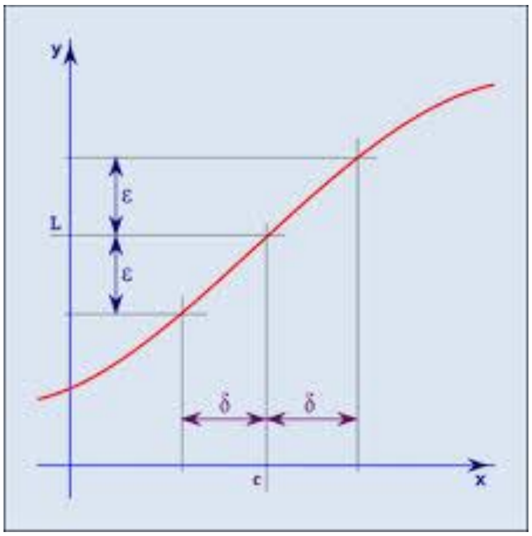
- Check CASA calendar for due dates.
- Bring “blank notes” to class. Completed notes will be posted after class.
- Do your best to attend every lecture and lab. This is a 4 credit course because of the lab component.
- Study after every lecture; work on the quiz covering the topic we cover on the lecture immediately afterwards. Retake your quizzes for more practice.
- Get help when you need help; bring your questions to the labs, or my office hours. We also have tutoring options on campus.
- Make sure you are a member of our team; check the discussion channel for announcements. You can post questions there. Make sure MS teams notifications are ON so that you are notifies when we make announcements there.
- Respect your friends in class; stay away from distractive behavior. Do your best to concentrate on the lecture.
- If you email me, mention the course code in the subject line.

Section 1.3 – Definition of Limit

"Given any $\varepsilon > 0$; we want to make $|f(x) - L|$ less than ε by choosing a sufficiently small δ and requiring that $0 < |x - c| < \delta$."

This statement may sound strange; here's the picture of what we're trying to accomplish:



Definition: The Limit of a Function

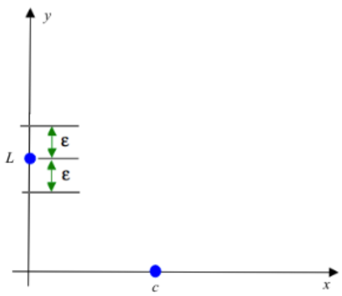
Let f be a function defined on an interval $(c-r, c) \cup (c, c+r)$ (where $r > 0$).

We say that the real number L is the limit of $f(x)$ if and only if

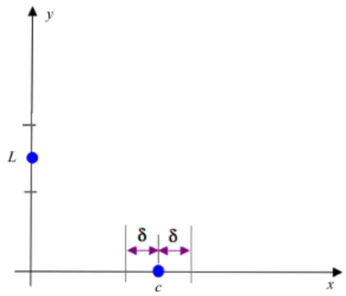
for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

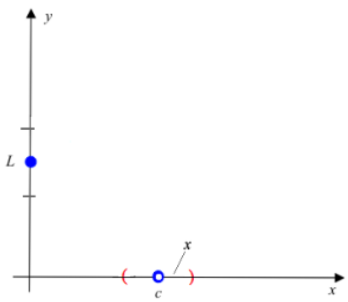
This is symbolized as $\lim_{x \to c} f(x) = L$.



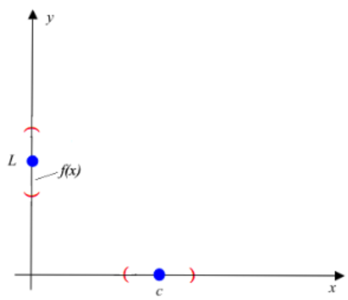
For each $\varepsilon > 0$



there exists $\delta > 0$ such that,



if $0 < |x - c| < \delta$,



then $|f(x) - L| < \varepsilon$.

Check this link: https://www.geogebra.org/m/msqdhy6g

Exercise: Show that $\lim_{x \to 4} (2x - 3) = 5$.

Proof:

Let $\varepsilon > 0$. Choose $\delta = \dfrac{\varepsilon}{2}$.

For $0 < |x - 4| < \delta$, we have $|f(x) - L| = |(2x - 3) - 5| = |2x - 8| = 2|x - 4|$.

That is, $|f(x) - L| = 2|x - 4| < 2\delta = \varepsilon$

Hence, $|f(x) - L| < 2\varepsilon$ whenever $0 < |x - 4| < \delta$.

We conclude that $\lim_{x \to 4} (2x - 3) = 5$.

End of proof.

Theorem 1.3.1: The uniqueness of limit

If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} f(x) = K$, then $L = K$.

That is, if the limit exists, it is unique.

Theorem 1.3.2: Arithmetic Rules

If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then:

(i) $\lim_{x\to c}\left[f(x) \pm g(x)\right] = L \pm M$.

That is, the limit of the sum of two functions is the sum of the limits.

(ii) $\lim_{x\to c}\left[kf(x)\right] = kL$, for each number k,

(iii) $\lim_{x\to c}\left[f(x)g(x)\right] = LM$.

That is, the limit of the product of two functions is the product of the limits.

(iv) If $\lim_{x\to c} f(x) = L$ with $L \neq 0$, then $\lim_{x\to c} \frac{1}{f(x)} = \frac{1}{L}$.

(v) If $M \neq 0$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$.

That is, the limit of a quotient is the quotient of the limits provided that the denominator does not tend to 0.

Remark: These facts are also valid for one sided limits; same rules can be written for limit as $x \to c^+$ or as $x \to c^-$.

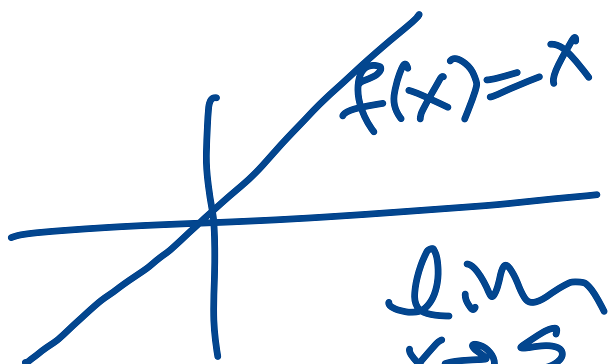
We want to start with some easy functions.

ex $f(x) = 1 \qquad \lim_{x\to 5} f(x) = 1$

Two specific rules:

1. Let k be a constant number. Then, for any c, $\lim_{x\to c} k = k$.

2. For any c, $\lim_{x\to c} x = c$



$\lim_{x\to 5} f(x) = 5$

These rules will be used for finding the limit of a polynomial.

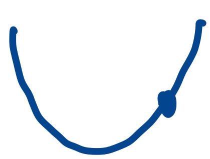
Example: ***(A typical polynomial)***

Evaluate $\lim_{x\to 2}\left(5x^2 - 4x + 7\right)$ if it exists.

poly.

$= \overbrace{5\cdot 2^2 - 4\cdot 2 + 7}^{f(2)}$

$= 20 - 8 + 7 = \boxed{19}$



$\lim_{x\to c}$ (formula)

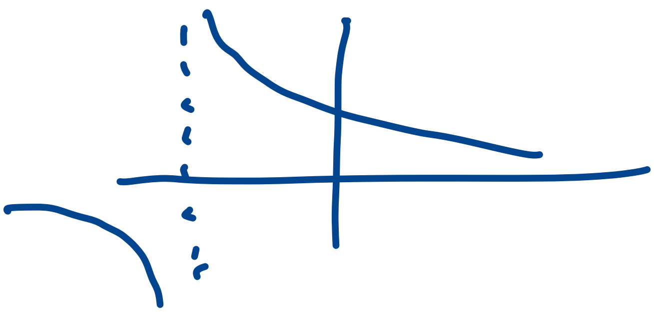
Fact: For any polynomial $p(x)$ and real number c,

$$\lim_{x \to c} p(x) = p(c).$$

Example 4: ***(A rational function whose denominator does not approach 0)***

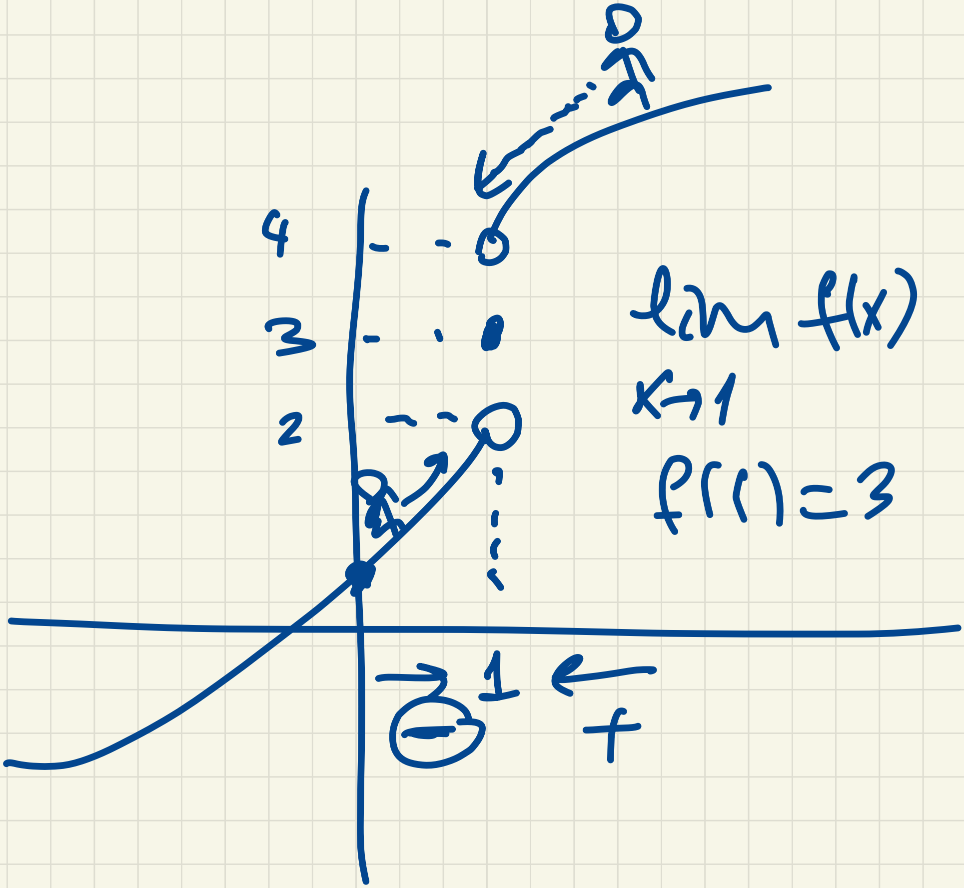
Evaluate $\lim_{x \to 1} \left(\dfrac{x^2+2}{x+6} \right)$ if it exists.

$$= \frac{1^2+2}{1+6} = \boxed{\frac{3}{7}}$$



Fact: If $p(x)$ and $q(x)$ are polynomials with $q(c) \neq 0$, then

$$\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}.$$



$$\lim_{x\to 1} f(x)$$

$$f(1)=3$$

$$\lim_{x\to 0} f(x) \qquad f(0)$$

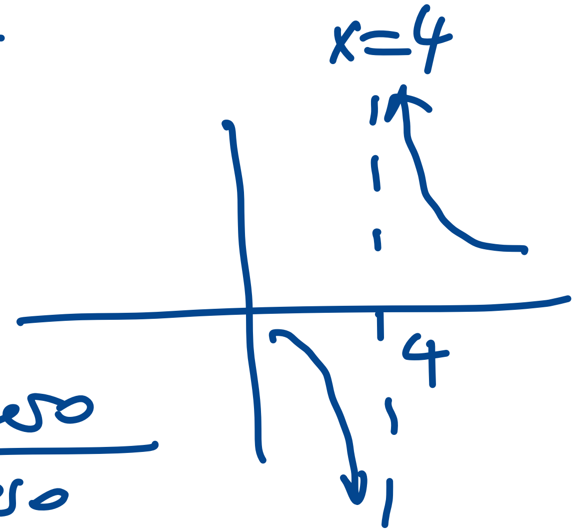
$$\lim_{x\to 2} f(x)$$

$$\lim_{x\to 10} f(x)$$

Example: (A rational function whose denominator approaches 0 while its numerator does not)

Evaluate $\lim\limits_{x\to4}\left(\dfrac{x^2}{x-4}\right)$ if it exists. $\leftarrow$ DNE

$$= \frac{4^2}{4-4} = \frac{16}{0}$$



$$\frac{\text{nonzero}}{\text{zero}}$$

What if? $\lim\limits_{x\to5}\left(\dfrac{x-5}{x^2-25}\right)$

$$= \frac{0}{0} \;?$$

work harder!

$$\lim_{x\to5}\left(\frac{\cancel{(x-5)}}{\cancel{(x-5)}\cdot(x+5)}\right)$$

Recall:

$$\lim_{x\to5}\frac{1}{x+5} = \boxed{\frac{1}{10}}$$

$a^2-b^2=(a-b)(a+b)$

$a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$

$a^3-1=(a-1)\left(a^2+a+1\right)$

$a^3+b^3=(a+b)\left(a^2-ab+b^2\right)$

Facts:

If $\lim_{x\to c} f(x) \neq 0$ while $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c} \frac{f(x)}{g(x)}$ does not exist.

If $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c} \frac{f(x)}{g(x)}$ gives an ***"indeterminate form of type* 0/0"** and the limit may or may not exist.

> More work is needed to investigate this limit. This situation is where most of the interesting limit problems are.

Fact: If $f(x) = \varphi(x)$ for all x near but not equal to c, then

$$\lim_{x\to c} f(x) = \lim_{x\to c} \varphi(x).$$

Here, think of $\varphi(x)$ as a simplified version of $f(x)$ whose limit as $x \to c$ is easy to determine, such as a polynomial or a rational function whose numerator and denominator do not both approach 0.

Summary: In general, to find the limit of a function defined with a formula, we should first try direct substitution.

- If we get a real number, then the limit is $f(c)$.
- If we get a fraction that looks like $\frac{a}{0}$ (where $a \neq 0$), then the limit does not exist.

If we get a fraction of the form $\frac{0}{0}$ (this is called an "indeterminate form of type 0/0"), that means we need to work harder to figure out what the limit is. Try simplifying the given function. This can be done by algebraic methods such as factoring (and canceling common factors), rationalizing, or using some other techniques.

Example: Find $\lim_{x\to 5}\left(\dfrac{x-5}{x^2-25}\right)$.

$x^3 - 1^3$

Example: Find $\lim_{x\to 1}\left(\dfrac{x^3-1}{\underbrace{x^2-5x+4}_{\text{quad.}}}\right)$ $= \dfrac{1^3-1}{1^2-5\cdot 1+4} = \dfrac{0}{0}$

$x \to 1$

work harder!

$= \lim_{x\to 1} \dfrac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}\cdot(x-4)}$

$= \lim_{x\to 1}\left(\dfrac{x^2+x+1}{x-4}\right) = \dfrac{1^2+1+1}{1-4} = \dfrac{3}{-3}$

plug in $x=1$

$= \boxed{-1}$

Similar to WHW 1

Example: Evaluate $\lim_{x\to 16}\left(\frac{x-16}{2\sqrt{x}-8}\right)$ if it exists. $= \frac{16-16}{2\sqrt{16}-8} = \frac{0}{0}$ work harder

$2\sqrt{x}-8 = 2(\sqrt{x}-4)$

$$= \lim_{x\to 16} \frac{(x-16)\cdot(\sqrt{x}+4)}{2(\sqrt{x}-4)\cdot(\sqrt{x}+4)}$$

$$= \lim_{x\to 16} \frac{\cancel{(x-16)}\cdot(\sqrt{x}+4)}{2\cdot\cancel{(x-16)}}$$

$(x-y)\cdot(x+y) = x^2-y^2$

$$= \lim_{x\to 16} \frac{\sqrt{x}+4}{2}$$

$(\sqrt{x}-\sqrt{y})\cdot(\sqrt{x}+\sqrt{y}) = x-y$

$$= \frac{\sqrt{16}+4}{2} = \frac{8}{2} = \boxed{4}$$

$$\frac{5\cdot(\sqrt{7}+1)}{(\sqrt{7}-1)(\sqrt{7}+1)}$$

Example: Evaluate $\lim_{h\to 0}\left(\frac{\frac{1}{4}-\frac{1}{h+4}}{h}\right) = ?$ if it exists.

$$= \frac{\frac{1}{4}-\frac{1}{4}}{0} = \frac{0}{0} \quad \text{work harder!}$$

$$\frac{1}{4}_{(h+4)} - \frac{1}{h+4}_{(4)} = \frac{h+4-4}{4(h+4)} = \frac{h}{4(h+4)}$$

rewrite cmplx. fr.

$$= \lim_{h\to 0} \frac{\frac{h}{4(h+4)}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{4(h+4)\cdot\cancel{h}}$$

$$= \lim_{h\to 0} \frac{1}{4(h+4)} = \frac{1}{4\cdot 4} = \boxed{\frac{1}{16}}$$

$$\frac{\frac{A}{B}}{C} = \frac{A}{B\cdot C}$$

$$\frac{A}{B}\cdot\frac{1}{C} = \frac{A}{BC}$$

online Quiz 2

Example: Evaluate $\lim_{x\to 0}\dfrac{1-\dfrac{2}{x^2}}{5+\dfrac{4}{x^2}}$ if it exists.

$$=\lim_{x\to 0}\frac{\dfrac{x^2-2}{x^2}}{\dfrac{5x^2+4}{x^2}}$$

$$\frac{1}{1}-\frac{2}{x^2} \qquad (x^2)\quad(1)$$

$$\frac{5}{1}+\frac{4}{x^2} \qquad (x^2)\quad(1)$$

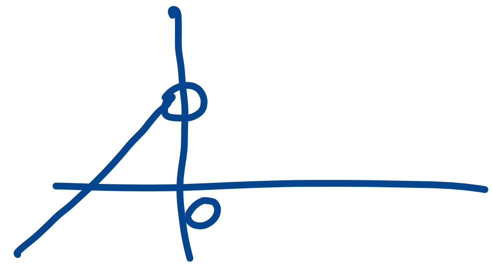
$$=\lim_{x\to 0}\frac{x^2-2}{\cancel{x^2}}\cdot\frac{\cancel{x^2}}{5x^2+4}$$

$$\frac{\dfrac{A}{B}}{\dfrac{C}{D}}=\frac{A}{B}\cdot\frac{D}{C}$$

$$=\lim_{x\to 0}\left(\frac{x^2-2}{5x^2+4}\right)$$

$$=\frac{0^2-2}{5\cdot 0^2+4}=\frac{-2}{4}=\boxed{\frac{-1}{2}}$$

Finally, how do we work with a piecewise function?



Example: (A piecewise function)

Consider the function: $f(x)=\begin{cases} x+2, & \text{if } x<0 \\ x^2+2, & \text{if } 0\le x<5 \\ x-1, & \text{if } 5\le x \end{cases}$

Evaluate

a) $\displaystyle\lim_{x\to 0} f(x)$,

RHL: $\displaystyle\lim_{x\to 0^+}(x^2+2)=0^2+2=2$

$x>0$

LHL: $\displaystyle\lim_{x\to 0^-}(x+2)=0+2=2$

$x<0$

$\displaystyle\lim_{x\to 0} f(x)=2$

b) $\displaystyle\lim_{x\to 5} f(x)$,

c) $\displaystyle\lim_{x\to 4} f(x)$.

b) $\displaystyle\lim_{x\to 5^+}(x-1)=5-1=4$

$x>5$

$\neq$

$\displaystyle\lim_{x\to 5^-}(x^2+2)=5^2+2=27$

$\displaystyle\lim_{x\to 5} f(x)$: DNE

c) $\displaystyle\lim_{x\to 4} f(x)=4^2+2=\underline{\underline{18}}$

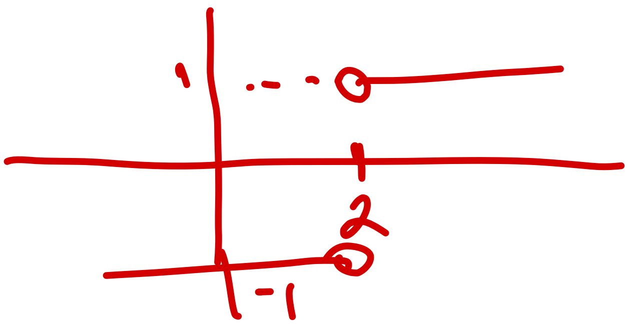
Example: Evaluate $\lim_{x\to 2}\frac{x-2}{|x-2|}$ if it exists.

$f(x) = \frac{x-2}{|x-2|}$

$\lim_{x\to 2^+}\frac{(x-2)}{(x-2)} = 1$

$x > 2$

$x - 2 > 0$



$\lim_{x\to 2^-} f(x) = -1$

$\lim_{x\to 2} f(x)$: DNE

Exercise: $f(x) = \begin{cases} \frac{x^2-5}{x-5}, & \text{if } x \neq 5 \\ 4, & \text{if } x = 5 \end{cases}$

Find: $\lim_{x\to 5} f(x)$

$\lim_{x\to 5^+}(x+5) = 10$

$\lim_{x\to 5^-}(x+5) = 10$

$= \checkmark$ $\boxed{\lim_{x\to 5} f(x) = 10}$

* $f(5) = 4$ by definition.

* if $x \neq 5$, $\frac{x^2-5}{x-5} = x+5 = f(x)$

$\lim_{x\to\infty} f(x)$

LIMIT AS x approaches Infinity

OR $\lim_{x\to-\infty} f(x)$

Notation: If the function values approach a real number L as x increases without bound, then we write: $\lim_{x\to\infty} f(x) = L$.

If the function values approach a real number M as x decreases without bound, then we write: $\lim_{x\to-\infty} f(x) = M$.

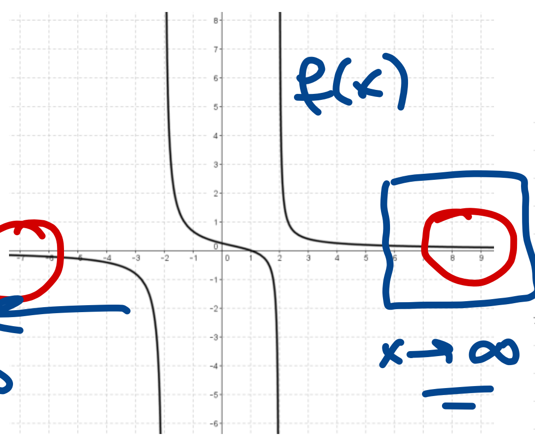
If $\lim_{x\to\infty} f(x) = L$, then the line $y = L$ is a (rightward) horizontal .

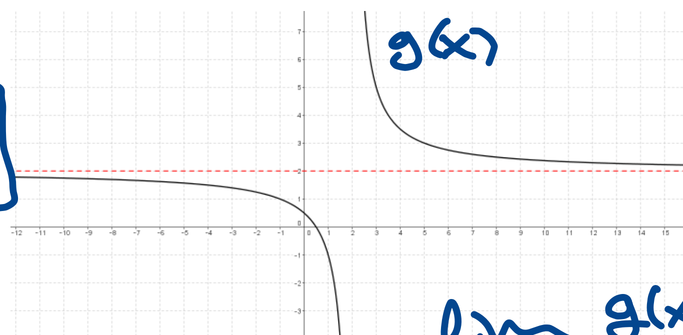
If $\lim_{x\to-\infty} f(x) = M$, then the line $y = M$ is a (leftward) horizontal asymptote.

If the graph is given, check the behavior of the given function as $x \to \infty$ or as $x \to -\infty$.

Examples:

HA $y=0$



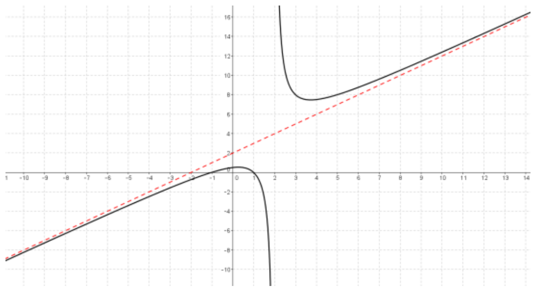


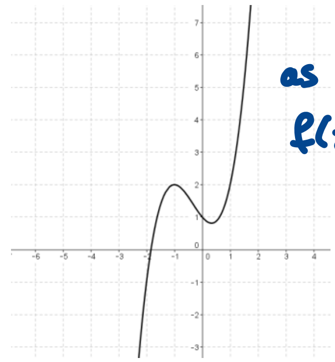
$y=2$

$\lim_{x\to\infty} f(x) = 0$

$\lim_{x\to\infty} g(x) = 2$

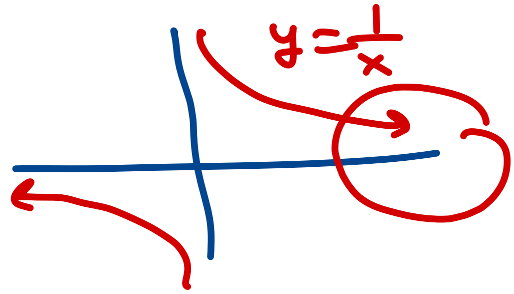
$\lim_{x\to-\infty} f(x) = 0$





as $x \to \infty$

$f(x) \to \infty$



$y = \frac{1}{x}$

What if the function is defined by a formula?

Example:

$$\lim_{x\to\infty}\left(\frac{1}{x}\right) = 0$$

as $x \to \infty$, $\frac{1}{x} \to 0$

$$\lim_{x\to\infty}\left(\frac{1}{x^2}\right) = 0$$

Fact: For $n > 0$ (a positive rational number) $\lim\limits_{x\to\infty}\left(\frac{1}{x^n}\right) = 0$.

Example:

$$\lim_{x\to\infty}\left(\frac{5x^2+1}{4x^2+x}\right)= \quad \frac{5}{4}$$

Remark: In general, there is an easier way to find these limits involving rational functions. We can use the methods covered in algebra about finding the horizontal asymptotes.

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\frac{\text{leading coefficient of f}}{\text{leadingcoefficient of g}}$$ if numerator and denominator have the same degrees.

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=0$$ if the denominator has bigger degree. (deg(N)<deg(D)).

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}:\text{does not exist}$$ if the numerator has bigger degree.

Example:

$$\lim_{x\to\infty}\left(\frac{x^2-x+1}{x^3+2}\right)= \quad 0$$

Example:

$$\lim_{x\to\infty}\left(\frac{x+x^4}{x^3-2x^4}\right)= \quad \frac{1}{-2} = -\frac{1}{2}$$

Example:

$$\lim_{x\to-\infty}\left(\frac{x+x^5}{x^3-2x}\right)= \quad \text{DNE}$$

Recall that polynomials do not have horizontal asymptotes.

Example:

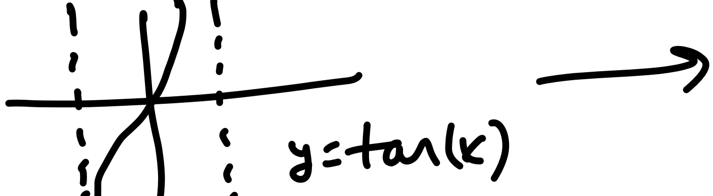
$$\lim_{x\to\infty}\left(2x^3+x-1\right)= \quad \text{DNE}$$

as $x\to\infty$

$f(x)\to\infty$

$$\lim_{x\to\infty}\left(-5x^4+x\right)= \quad \text{DNE}$$

as $x\to\infty$, $f(x)\to-\infty$.

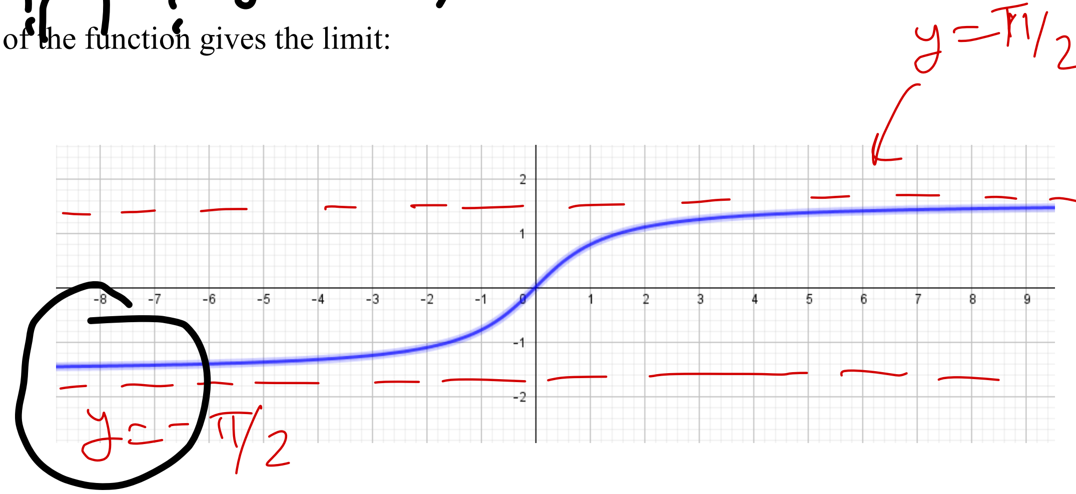


Many times, a quick sketch of the function gives the limit:

Example:

$\lim_{x\to\infty}\left(\arctan(x)\right) =$

$= \frac{\pi}{2}$

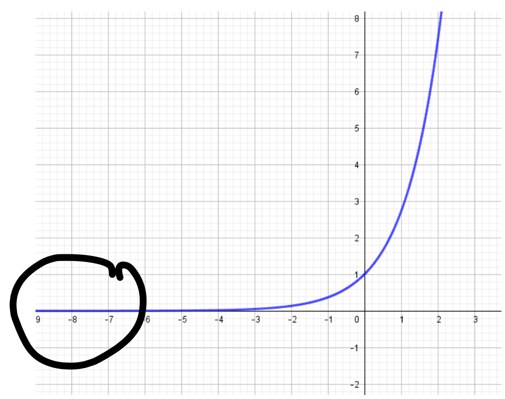


$\lim_{x\to-\infty}\left(\arctan(x)\right) =$

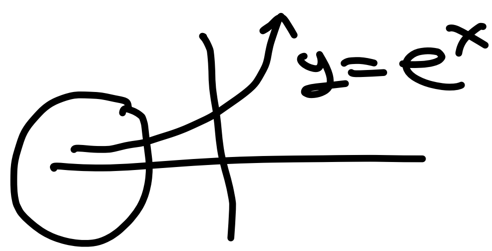
$= -\frac{\pi}{2}$

$\lim_{x\to\infty}\left(e^{x}\right) =$

as $x \to \infty$, $e^x \to \infty$



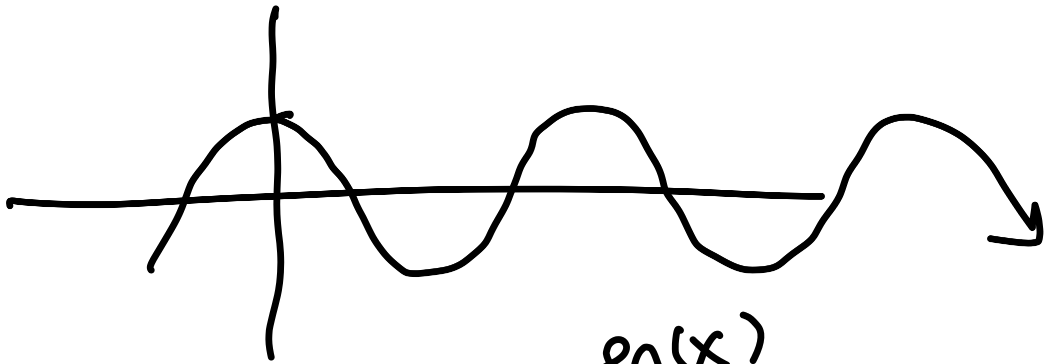
$\lim_{x\to-\infty}\left(e^{x}\right) = 0$



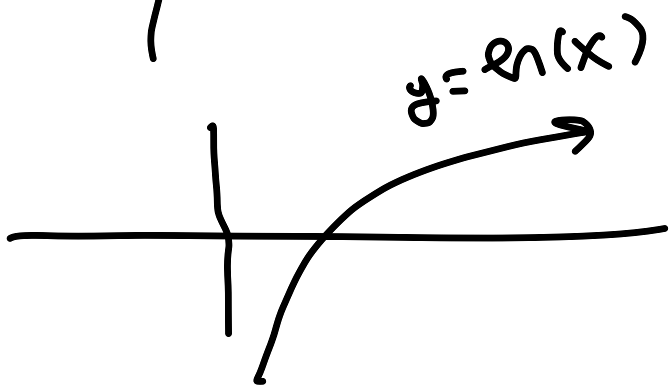
Exercise: Answer the following questions; use the "graphs" of given questions to find the answers.

$\lim_{x\to\infty} (\cos(x)) =$

$\lim_{x\to\infty} (\sin(x)) =$



$\lim_{x\to\infty} \left(e^{2x}\right) =$



$\lim_{x\to\infty} (\ln(x)) =$ → ∞

What if? $\lim_{x\to\infty} \left(\frac{1}{\ln(x)}\right) = ?$ 0

Homework #1 is posted on CASA; due in lab.

Work on your homework right after class, don't wait until the last day. Take your quizzes right after we finish a topic in class. Retake them – practice makes perfect! Due dates are the last days to retake a quiz – not the day of your first attempt.

Read Sections 1.2, 1.3 from your textbook.

Check CASA calendar and our Team regularly for announcements.