Math 2413- Calculus I

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- Check CASA calendar for due dates.
- Bring "blank notes" to class. Completed notes will be posted after class.
- Do your best to attend every lecture and lab. This is a 4 credit course because of the lab component.
- Study after every lecture; work on the quiz covering the topic we cover on the lecture immediately afterwards. Retake your quizzes for more practice.
- Get help when you need help; bring your questions to the labs, or my office hours. We also have tutoring options on campus.
- Make sure you are a member of our team; check the discussion channel for announcements. You can post questions there. Make sure MS teams notifications are ON so that you are notifies when we make announcements there.
- Respect your friends in class; stay away from distractive behavior. Do your best to concentrate on the lecture.
- If you email me, mention the course code in the subject line.

Section 1.4 – Continuity

Definition: Let c be a real number and f be a function. We say that f is *continuous* at c if

$$\lim_{x \to c} f(x) = f(c).$$

The function is said to be discontinuous at c if it is not continuous there.

According to this definition, the function f is continuous at c if all of the following conditions are met:

1. f is defined at c,
2. $\lim_{x \to c} f(x)$ exists, and
3. $\lim_{x \to c} f(x) = f(c)$.

If any of these conditions fails, the function is not continuous at c. We may say that the function is **discontinuous** at c.

In other words, for *f* to be continuous at a point c, we need:

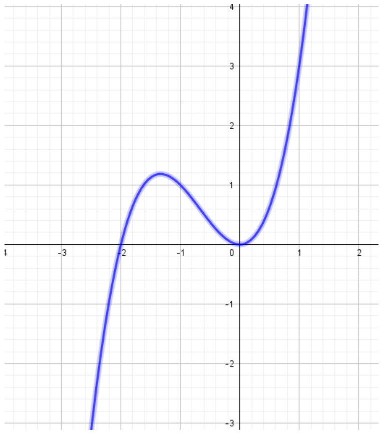
$$f(c) = \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)$$

(Function is defined, limit exists, they are equal)

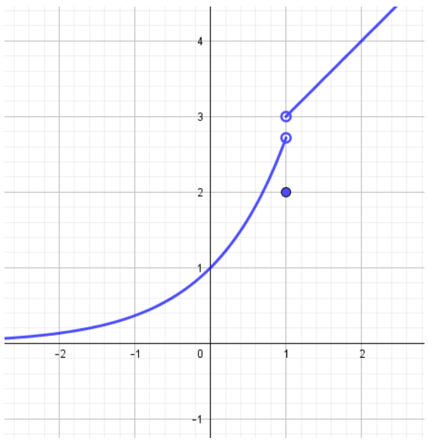
If ***f*** is not defined at *c*, or if the limit does not exist, the function is not continuous at *c*.

Geometrically speaking – a function is continuous if the graph has no holes or breaks. That is, you can trace the graph without removing your pen.

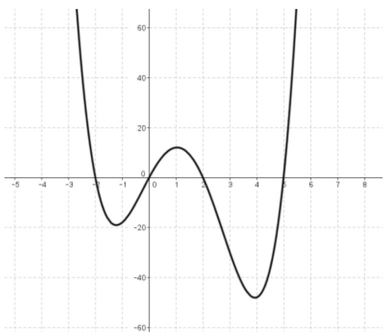
For example, the following function is continuous everywhere:

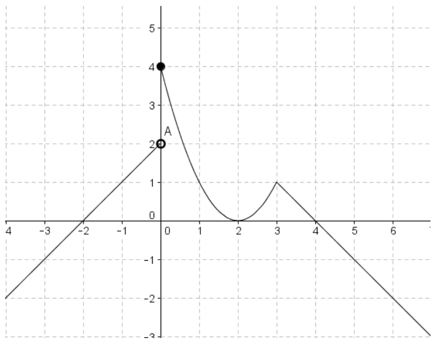


However, the following function is NOT continuous at a point:



Are the following functions continuous?





When is a function discontinuous at c?

1) If f is not defined at c, we know that graph has a hole or an asymptote at c and the function is not continuous there.

2) If f is defined at c (that is, c is in the domain of f), then f can be discontinuous for one of these reasons:

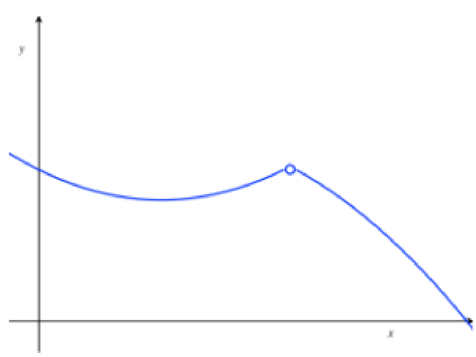
a) $\lim\limits_{x \to c} f(x)$ does not exist,

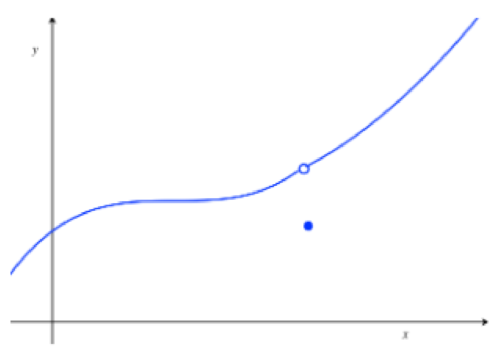
b) $\lim\limits_{x \to c} f(x)$ exists, but $\lim\limits_{x \to c} f(x) \neq f(c)$.

Types of discontinuity:

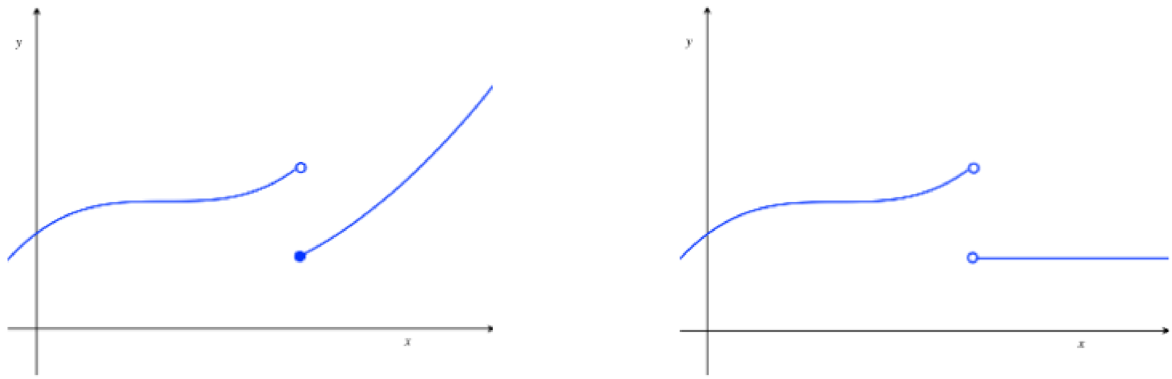
If a function f is discontinuous at c, this discontinuity can be classified as:

- *Removable discontinuity* if
  - (i) f is not defined at c, or
  - (ii) $f(c)$ and $\lim\limits_{x \to c} f(x)$ exist, but $\lim\limits_{x \to c} f(x) \neq f(c)$.

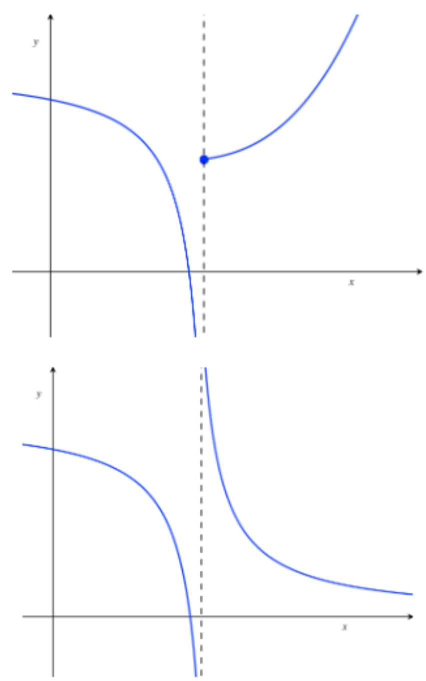




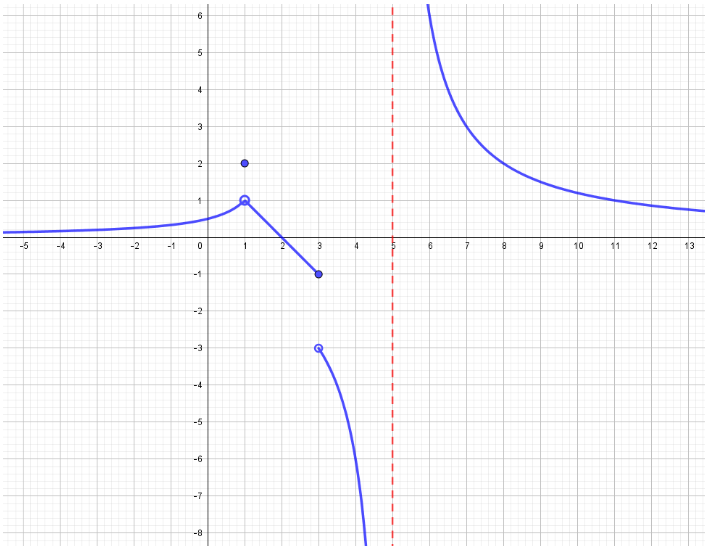
- *Jump discontinuity* if each one-sided limit exists but they are not equal.



- *Infinite discontinuity* if $f(x) \to \pm\infty$ on at least one side of c. This type is generally associated with having a vertical asymptote at $x = c$.



Example: Study the continuity of the function and classify each point of discontinuity as jump, removable, or infinite.



Identify the points to study: ______________________________________

Study possible points of discontinuity:

$x = 1$:

Work: Function Value:

RHL: $\lim_{x \to 1^+} f(x) =$ LHL: $\lim_{x \to 1^-} f(x) =$ Limit: $\lim_{x \to 1} f(x) =$

Conclusion: the function has _________ discontinuity at $x = 1$.

$x = 3$:

Work: Function Value:

RHL: LHL: Limit: $\lim_{x \to 3} f(x) =$

Conclusion: the function has _________ discontinuity at $x = 0$.

$x = 5$:

Work: Function has a vertical asymptote at $x = 5$.

Conclusion: the function has _________ discontinuity at $x = 5$.

What if the function is defined by a formula?

Fact: The following types of functions are continuous at every number **in their domains**:

- Polynomials,
- Rational functions,
- Root functions,
- Trigonometric functions,
- Inverse trigonometric functions,
- Exponential functions,
- Logarithmic functions.

Polynomials

Fact: Polynomials are continuous everywhere.

Example: Find the points of discontinuity (if any): $f(x) = x^3 + x^2 - 1$.

Rational Functions

- **A rational function has infinite discontinuity at each vertical asymptote.**

Any number that makes the denominator equal to zero (while the top is not zero) is a Vertical Asymptote for the function. Each VA is an infinite type discontinuity.

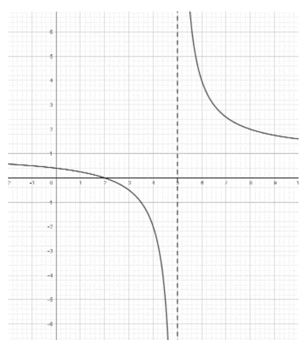
$$f(x) = \frac{x-a}{(x-b)(x-c)} \xrightarrow{\text{no common factors to cancel}} \text{V.A. at } x=b, x=c.$$

$\rightarrow$ infinite discontinuities at x=b and x=c.

Example: $f(x) = \frac{x-2}{x-5}$ has a vertical asymptote at $x = 5$.

Hence: This function has an infinite discontinuity at $x = 5$

Graph of $f(x) = \frac{x-2}{x-5}$:



- **A rational function might have a "hole"; a hole is a removable discontinuity.**

To figure out if the function has a hole, factor top and bottom completely. Any common factor than can be canceled creates a hole on the graph.

If: $f(x) = \frac{x-a}{(x-a)(x-b)} \xrightarrow{\text{cancel (x-a)}} f(x) = \frac{1}{(x-b)} \rightarrow \text{hole at } x=a, \text{ V.A. at } x=b.$

If: $f(x) = \frac{x-a}{(x-b)(x-c)} \xrightarrow{\text{no common factors to cancel}} \text{V.A. at } x=b, x=c.$

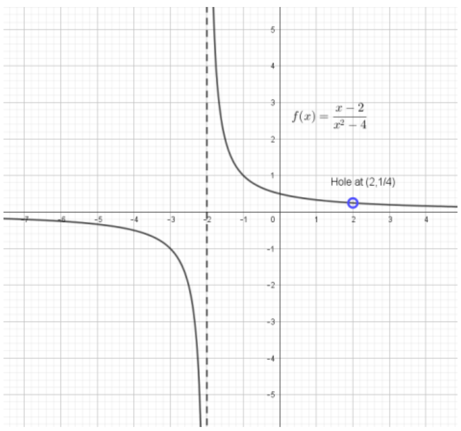
Example: $f(x)=\dfrac{x-2}{x^2-4}$ has a hole at $x=2$ since:

$$f(x)=\frac{x-2}{x^2-4}=\frac{x-2}{(x-2)(x+2)}\xrightarrow{\text{cancel (x-2)}} f(x)=\frac{1}{(x+2)}$$;a hole at $x=2$, a VA at $x=$ -2.

Hence, this function has a removable discontinuity at $x=2$.

There is a vertical asymptote at $x=-2$, so this function has an infinite discontinuity at $x=-2$.

See the Graph of $f(x)=\dfrac{x-2}{x^2-4}$:



Conclusion: The function $f(x)=\dfrac{x-2}{x^2-4}$ has two points of discontinuity;

It has a removable discontinuity at $x=2$, it has an infinite discontinuity at $x=-2$.

Example: When is $f(x) = \frac{x}{x^2 - 4x}$ discontinuous? Classify the points of discontinuity.

This function has a(n) ____________________ discontinuity at x = ________

This function has a(n) ____________________ discontinuity at x = ________

Multiple Choice Question:

Example: Given: $f(x) = \frac{x-5}{x^3 - 5x^2}$. Which of the following is a true statement about this function?

A) The function has infinite discontinuity at $x = 0$ and at $x = 5$.
B) The function has infinite discontinuity at $x = 0$, and a removable discontinuity at $x = 5$.
C) The function has infinite discontinuity at $x = 5$, and a removable discontinuity at $x = 0$.
D) The function has infinite discontinuity at $x = 0$, and a jump discontinuity at $x = 5$.
E) The function has no points of discontinuity.
F) None of the above.

Trigonometric functions

Note: Sine and Cosine functions are continuous everywhere. Tangent and cotangent functions have vertical asymptotes; so, they have points of discontinuity (infinite type).

Example: Find the points of discontinuity (if any): $f(x) = \sin(2x)$.

Example: Find the points of discontinuity (if any): $f(x) = \cos(x)$.

Multiple Choice Question:

Example: Which of the following is true about $f(x) = \tan(x)$?

A) The function has an infinite discontinuity at $x = 0$.
B) The function has an infinite discontinuity at $x = \frac{\pi}{2}$.
C) The function has an infinite discontinuity at $x = \frac{\pi}{4}$.
D) The function has no points of discontinuity.
E) None of the above.

Many complicated continuous functions can be built up using simple ones.

Theorem 1.4.1: If f and g are continuous at c, then

(i) $f + g$ is continuous at c,

(ii) $f - g$ is continuous at c,

(iii) kf is continuous at c (where k is any real number),

(iv) fg is continuous at c,

(v) f / g is continuous at c, provided $g(c) \neq 0$.

Parts (i) – (iv) can be extended to any finite number of functions.

Example: Find the points of discontinuity (if any): $f(x) = x^2 + \sin(x)$.

Example: Find the points of discontinuity (if any): $f(x) = \dfrac{2x}{\sin(x)}$.

Example: Find the points of discontinuity (if any): $f(x) = \dfrac{2x}{1-\cos(x)}$.

Example: If f and $f+g$ are continuous functions; which of the following statements is true?

A) $\dfrac{f}{f+g}$ is continuous.

B) g is continuous.

C) $\dfrac{f}{g}$ is continuous.

D) None of the above.

We studied one-sided limits in Section 1.2; similarly, we may consider one-sided continuity.

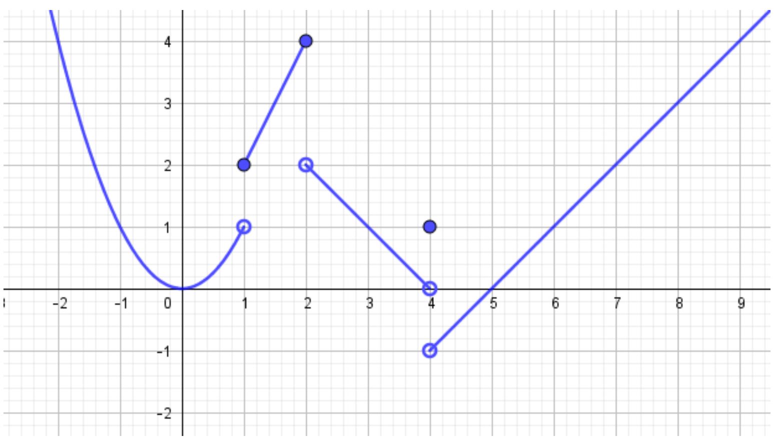
Definition:

A function f is said to be *continuous from the left at* c if $\lim_{x \to c^-} f(x) = f(c)$.

f is said to be *continuous from the right at* c if $\lim_{x \to c^+} f(x) = f(c)$.

f is continuous at *c* if it is continuous both from the right and left at *c*.

Example: Consider the function whose graph is given below.



Fill in the blanks:

The function is continuous from ________ at $x = 1$.

The function is continuous from ________ at $x = 2$.

The function is _______________________ at $x = 4$.

Continuity over an interval

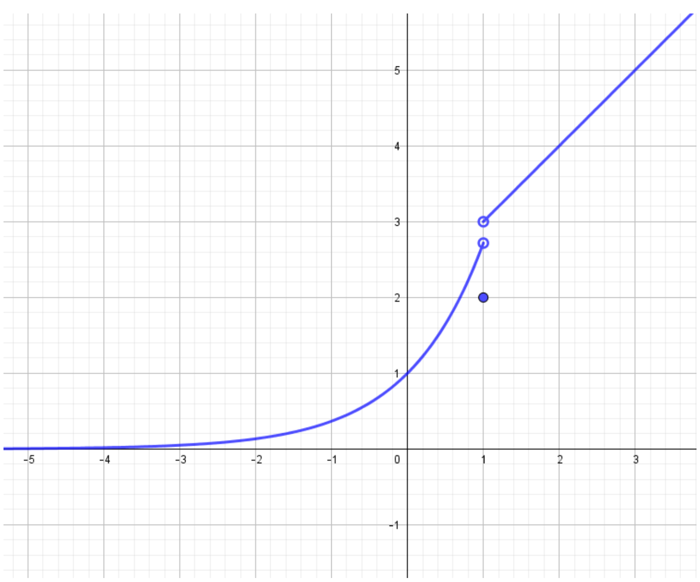
Definition: Let (a,b) be an open interval. A function is said to be *continuous over* (a,b) if it is continuous at every number in this interval.

If f is defined on a closed interval $[a,b]$, we only expect to have one-sided continuity at the end points a and b. That is, if the function is continuous at every number in (a,b), continuous from the right at a and continuous from the left at b, then we say that the function is continuous over $[a,b]$.

Example: The function $f(x)$ is graphed below.

This function is NOT continuous AT: _______ (point(s))

This function is continuous ON: ___________ (interval(s))



Example: Find the interval(s) over which the function $f(x) = x^2 + 2$ is continuous.

Answer: __________

Example: Find the interval(s) over which the function $f(x) = \sqrt{x-5}$ is continuous.

Answer: __________

Example: Find the interval(s) over which the function $f(x) = \frac{x-5}{(x-2)(x-4)}$ is continuous.

Answer: ____________________

How to work with piece-wise functions:

Identify possible points of discontinuity (breaks, vertical asymptotes, etc.)

Check each using the 3 steps (function value, RHL, LHL)

Recall: For f to be continuous at c, we need: $f(c) = \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)$

Example: Find all points of discontinuity and classify them:

$$f(x) = \begin{cases} x^2 + 6, & \text{if } x < 1 \\ 5x, & \text{if } 1 \le x \end{cases}$$

Points to investigate: ____________________________

$x =$ _____ Compare the two-sided limits and function value:

Function Value:

RHL:

LHL:

Conclusion: The function ______________________________

Example: Find all points of discontinuity and classify them:

$$f(x)=\begin{cases} x-1, & \text{if } x<4 \\ \dfrac{12}{x}, & \text{if } 4\le x<6 \\ \dfrac{10}{x-8}, & \text{if } x>6 \end{cases}$$

Points to investigate: ____________________________

$x=$ _____ Compare the two-sided limits and function value:

Function Value:

RHL:

LHL:

Conclusion:

$x=$ _____

Function Value:

RHL:

LHL:

Conclusion:

$x=$ _____

Function Value:

RHL:

LHL:

Conclusion:

Example: Find the values of A and B so that the function is continuous everywhere.

$$f(x)=\begin{cases} Ax^2-14, & \text{if } 2<x \\ 10, & \text{if } x=2 \\ Bx, & \text{if } x>2 \end{cases}$$

Example: The function $f(x)=\dfrac{x^2-1}{x-1}$ is continuous everywhere except for $x=1$.

Redefine this function so that it is continuous everywhere.

Redefine: $f(x)=\begin{cases} \dfrac{x^2-1}{x-1}, & \text{if } x\neq 1 \\ ??, & \text{if } x=1 \end{cases}$

Exercise: Find the value of A so that the function is continuous everywhere.

$$f(x)=\begin{cases} Ax^2, & \text{if } 5<x \\ 2x+A, & \text{if } x\geq 5 \end{cases}$$

Exercise: Find all points of discontinuity and classify them:

$$f(x)=\begin{cases} \dfrac{1}{x}, & \text{if } x<1 \\ \sqrt{x}, & \text{if } 1<x<4 \\ x-1, & \text{if } 4<x\leq 5 \\ \dfrac{12}{x-2}, & \text{if } 5<x \end{cases}$$

Study the points: x=1, x=4,x=5. (No need to study x=2!)

Exercise: Find all points of discontinuity and classify them:

$$f(x)=\begin{cases} 2x, & \text{if } x<0 \\ \sqrt{x}, & \text{if } 0\leq x<1 \\ 2x-1, & \text{if } 1<x\leq 4 \\ \dfrac{2}{x-10}, & \text{if } 4<x \end{cases}$$

Study the points: x=0, x=1, x=4 AND x=10 (VA for the last function).

Exercise: Graph a function satisfying all of the properties below:

- The function has removable discontinuity at $x = -1$.
- The function has jump discontinuity at $x = 2$.
- The Function has infinity discontinuity at $x = 6$.
- $\lim_{x \to \infty} f(x) = 0$
- $\lim_{x \to -\infty} f(x) = -1$