completed

Math 2413- Calculus I

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- Check CASA calendar for due dates.
- Bring "blank notes" to class. Completed notes will be posted after class.
- Do your best to attend every lecture and lab. This is a 4 credit course because of the lab component.
- Study after every lecture; work on the quiz covering the topic we cover on the lecture immediately afterwards. Retake your quizzes for more practice.
- Get help when you need help; bring your questions to the labs, or my office hours. We also have tutoring options on campus.
- Make sure you are a member of our team; check the discussion channel for announcements. You can post questions there. Make sure MS teams notifications are ON so that you are notifies when we make announcements there.
- Respect your friends in class; stay away from distractive behavior. Do your best to concentrate on the lecture.
- If you email me, mention the course code in the subject line.

HW : Solve the exercises on the last page!!!

Week 1:

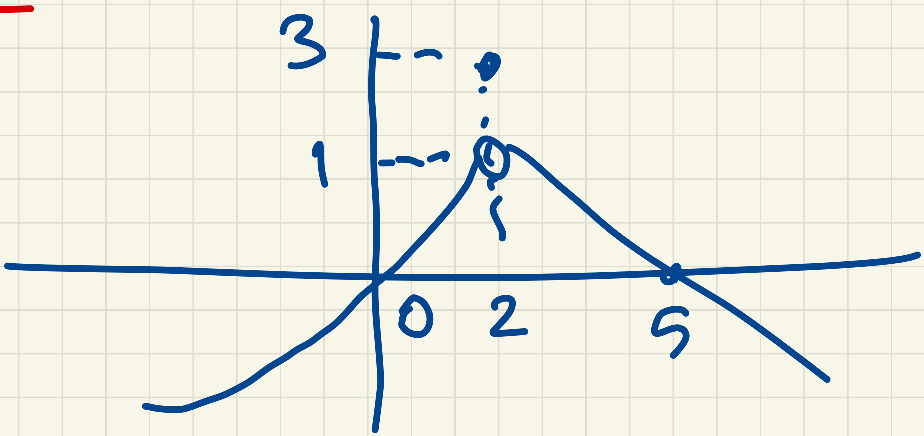
Covered 1.2 & 1.3

WHW1 due in class today

WHW2 due next Tue.

Lab Quiz 2 covers 1.3 & 1.4
(Thursday)

$$\lim_{x \to \infty} f(x)$$



Section 1.4 – Continuity

Definition: Let c be a real number and f be a function. We say that f is *continuous* at c if

$$\lim_{x \to c} f(x) = f(c).$$

The function is said to be discontinuous at c if it is not continuous there.

According to this definition, the function f is continuous at c if all of the following conditions are met:

1. f is defined at c,
2. $\lim_{x \to c} f(x)$ exists, and
3. $\lim_{x \to c} f(x) = f(c)$.

If any of these conditions fails, the function is not continuous at c. We may say that the function is **discontinuous** at c.

In other words, for *f* to be continuous at a point c, we need:

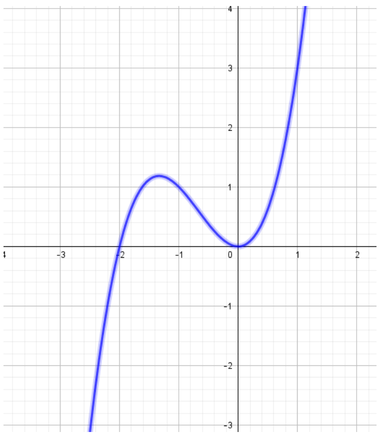
$$f(c) = \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)$$

(Function is defined, limit exists, they are equal)

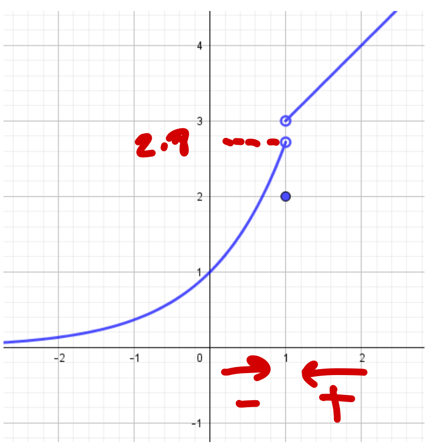
If ***f*** is not defined at *c*, or if the limit does not exist, the function is not continuous at *c*.

Geometrically speaking – a function is continuous if the graph has no holes or breaks. That is, you can trace the graph without removing your pen.

For example, the following function is continuous everywhere:



However, the following function is NOT continuous at a point:

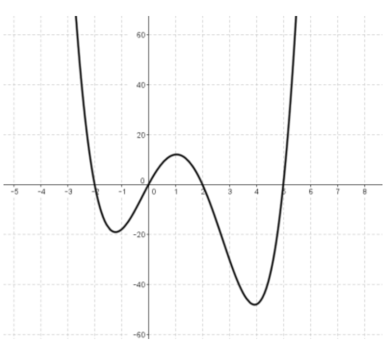


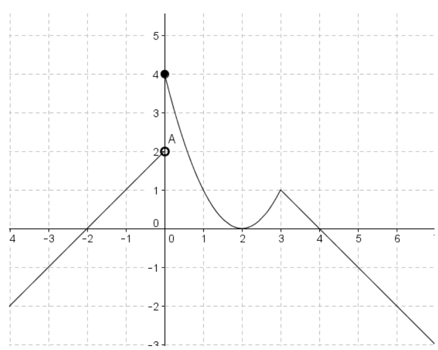
$f(1) = 2$

$\left.\begin{array}{l} \lim_{x \to 1^+} f(x) = 3 \\ \lim_{x \to 1^-} f(x) = 2.8 \end{array}\right] \neq$

$\lim_{x \to 1} f(x)$: DNE ✗

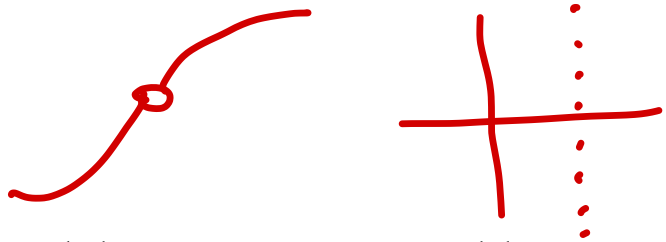
Are the following functions continuous?





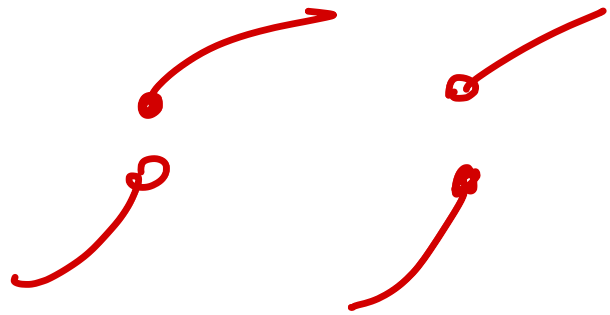
Discts at $x=0$.

When is a function discontinuous at c?



1) If f is not defined at c, we know that graph has a hole or an asymptote at c and the function is not continuous there.

2) If f is defined at c (that is, c is in the domain of f), then f can be discontinuous for one of these reasons:



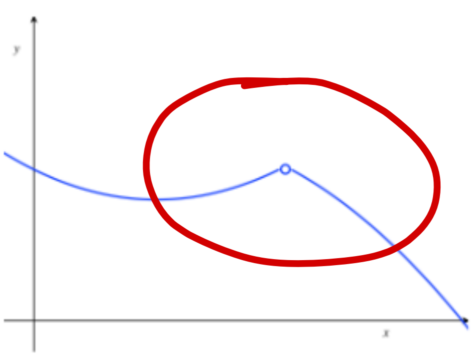
a) $\lim\limits_{x \to c} f(x)$ does not exist,

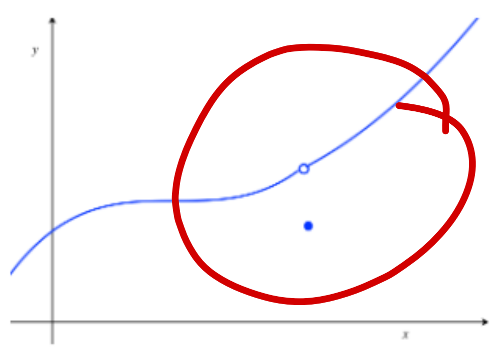
b) $\lim\limits_{x \to c} f(x)$ exists, but $\lim\limits_{x \to c} f(x) \neq f(c)$.

Types of discontinuity:

If a function f is discontinuous at c, this discontinuity can be classified as:

- *Removable discontinuity* if
  - (i) f is not defined at c, or
  - (ii) $f(c)$ and $\lim\limits_{x \to c} f(x)$ exist, but $\lim\limits_{x \to c} f(x) \neq f(c)$.

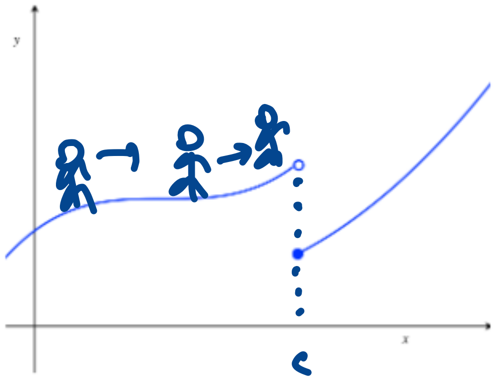


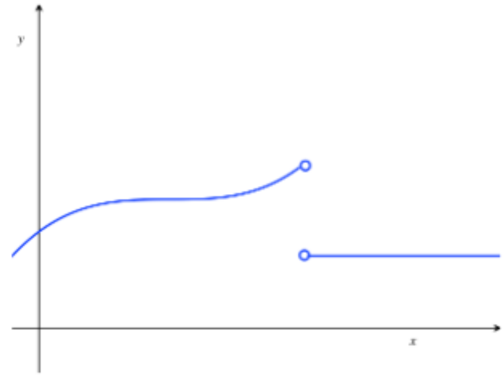


$f(c)$ is not defined
limit exists

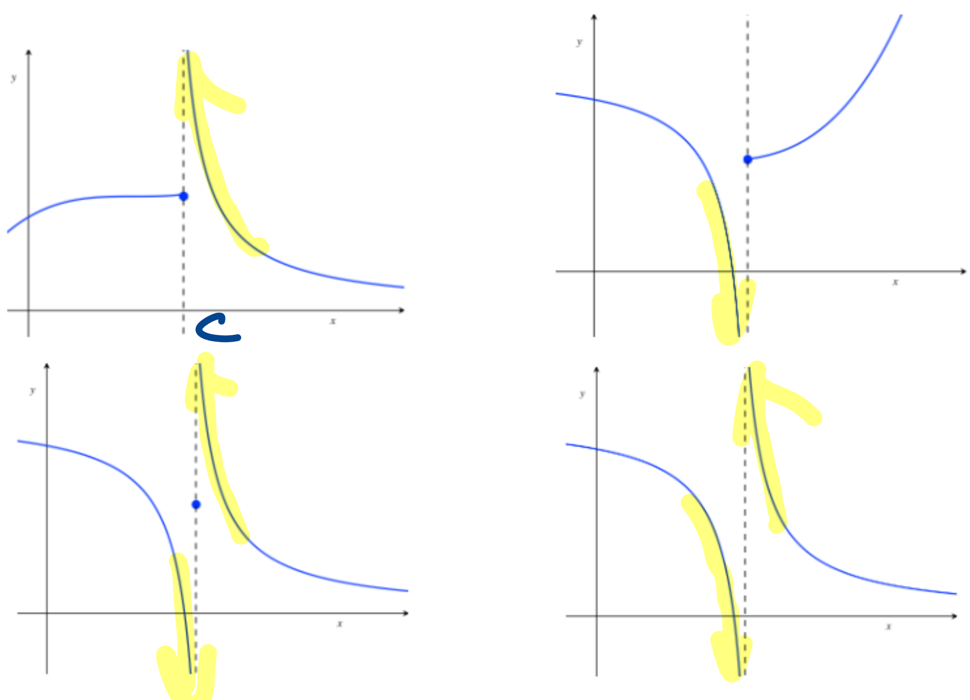
$f(c) \neq \lim\limits_{x \to c} f(x)$

- *Jump discontinuity* if each one-sided limit exists but they are not equal.

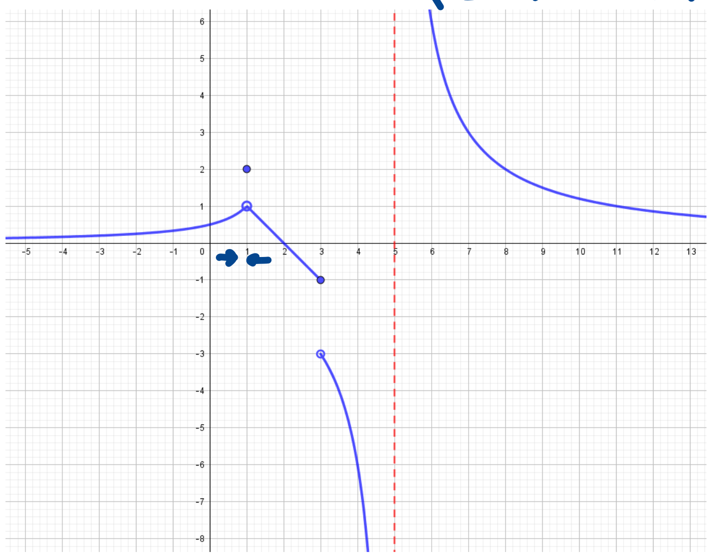




- *Infinite discontinuity* if $f(x) \to \pm\infty$ on at least one side of c. This type is generally associated with having a vertical asymptote at $x = c$.



Example: Study the continuity of the function and classify each point of discontinuity as jump, removable, or infinite. $x=5$ VA



Identify the points to study: $x=1$, $x=3$, $x=5$

Study possible points of discontinuity:

$x=1$:

Work: Function Value: $f(1)=2$

RHL: $\lim_{x\to 1^+} f(x)=1$ LHL: $\lim_{x\to 1^-} f(x)=1$ Limit: $\lim_{x\to 1} f(x)=1$

$\neq$

Conclusion: the function has removable discontinuity at $x=1$.

$x=3$:

Work: Function Value: $f(3)=-1$

RHL: -3 LHL: -1 Limit: $\lim_{x\to 3} f(x)=$ DNE

Conclusion: the function has jump discontinuity at $x=$ ~~6~~ 3.

$x=5$:

Work: Function has a vertical asymptote at $x=5$.

Conclusion: the function has infinite discontinuity at $x=5$.

$f(x)\to\infty$ as $x\to 5^+$

$f(x)\to -\infty$ as $x\to 5^-$

What if the function is defined by a formula?

Fact: The following types of functions are continuous at every number **<u>in their domains</u>**:

- Polynomials,
- Rational functions,
- Root functions,
- Trigonometric functions,
- Inverse trigonometric functions,
- Exponential functions,
- Logarithmic functions.

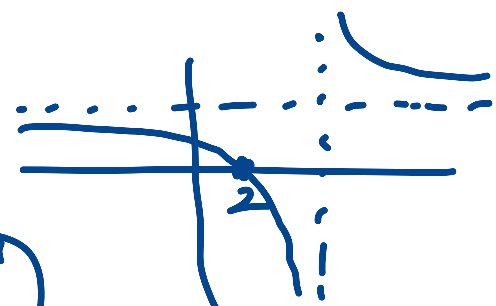
Polynomial

Example: Find the points of discontinuity (if any): $f(x) = x^3 + x^2 - 1$.

None

$f(2)=0$

$x=5$



Example: Find the points of discontinuity (if any): $f(x) = \frac{x-2}{x-5}$.

Domain: $(-\infty, 5) \cup (5, \infty)$

Discts at $x=5$ (infinite)

Example: Find the points of discontinuity (if any): $f(x) = \frac{x-2}{x^2-4}$.

$x=2$: removable discty

$$f(x) = \frac{\cancel{x-2}}{(\cancel{x-2})(x+2)} = \frac{1}{x+2}$$

$x=-2$: infinite discty.

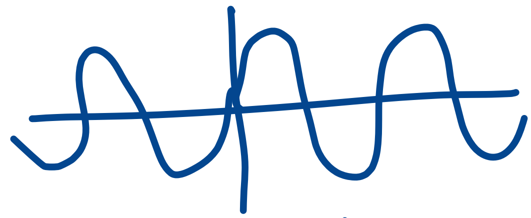
Hole at $x=2$

VA at $x=-2$

continuous on: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

Example: Find the points of discontinuity (if any): $f(x) = \sin(2x)$.

None



Example: When is $f(x) = \dfrac{x}{\underbrace{x^2 - x}_{x(x-1)}}$ discontinuous?

$x = 0 \leftarrow$ hole removable

$x = 1 \leftarrow$ VA infinite discts

Example: When is $f(x) = \cos(x)$ discontinuous?

Never.

When is $f(x) = \cos(x)$ continuous?

$(-\infty, \infty)$

Many complicated continuous functions can be built up using simple ones.

Theorem 1.4.1: If f and g are continuous at c, then

(i) $f + g$ is continuous at c,

(ii) $f - g$ is continuous at c,

(iii) kf is continuous at c (where k is any real number),

(iv) fg is continuous at c,

(v) f / g is continuous at c, provided $g(c) \neq 0$.

Parts (i) – (iv) can be extended to any finite number of functions.

$\underbrace{x^2} + \underbrace{\sin(x)}$ cts at $x=0$?

Example: Find the points of discontinuity (if any): $f(x) = \underbrace{x^2 + \sin(x)}$.

None

Example: Find the points of discontinuity (if any): $f(x) = \dfrac{2x}{\sin(x)}$.

$\sin(x) = 0$?

$x = 0, \pi, 2\pi, 3\pi, \ldots$

$\{\pm k\pi\}$

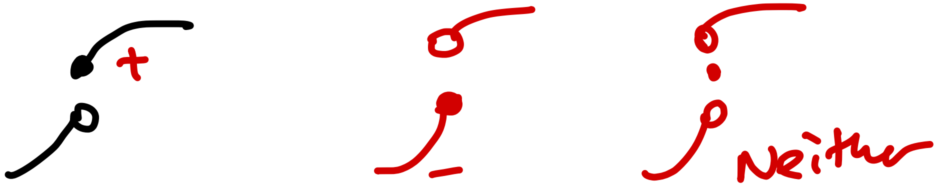
Example: Find the points of discontinuity (if any): $f(x) = \dfrac{2x}{1-\cos(x)}$.

$1 - \cos(x) = 0$

$\cos(x) = 1$

if: $x = 0, 2\pi, 4\pi, \ldots$

$\{\pm 2k\pi\}$



We studied one-sided limits in Section 1.2; similarly, we may consider one-sided continuity.

Definition:

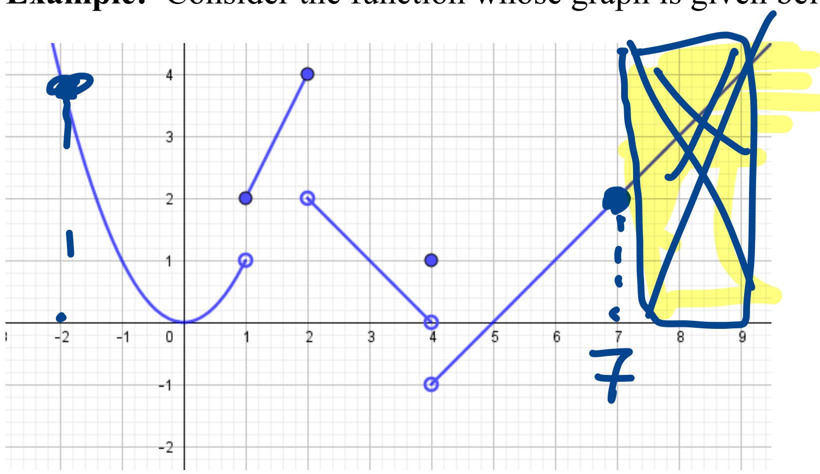
A function f is said to be *continuous from the left at* c if $\lim\limits_{x \to c^-} f(x) = f(c)$.

f is said to be *continuous from the right at* c if $\lim\limits_{x \to c^+} f(x) = f(c)$.

f is continuous at c if it is continuous both from the right and left at c.

Example: Consider the function whose graph is given below.

$f: [-2, 7] \to \ldots$



discts

Fill in the blanks:

The function is continuous from right at $x = 1$.

The function is continuous from left at $x = 2$.

The function is discts at $x = 4$.

Continuity over an interval

Definition: Let (a,b) be an open interval. A function is said to be *continuous over* (a,b) if it is continuous at every number in this interval.

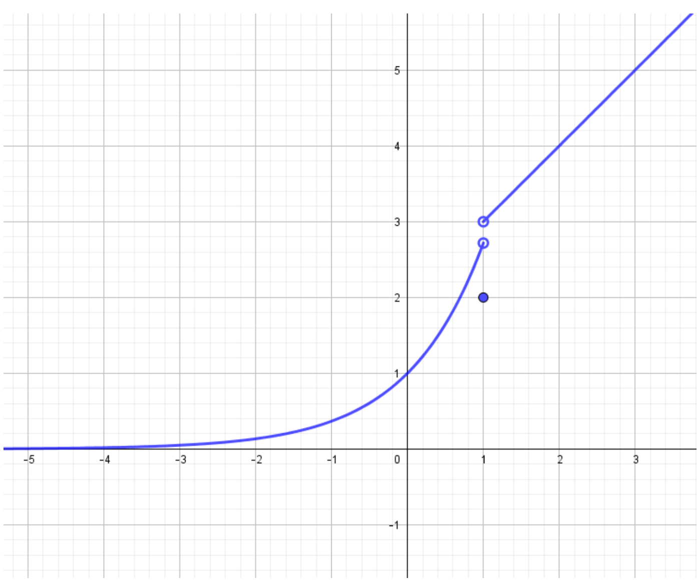
If f is defined on a closed interval $[a,b]$, we only expect to have one-sided continuity at the end points a and b. That is, if the function is continuous at every number in (a,b), continuous from the right at a and continuous from the left at b, then we say that the function is continuous over $[a,b]$.

Example: The function $f(x)$ is graphed below.

This function is NOT continuous AT: $x=1$ (point(s))

This function is continuous ON: ___________ (interval(s))

$(-\infty, 1) \cup (1, \infty)$



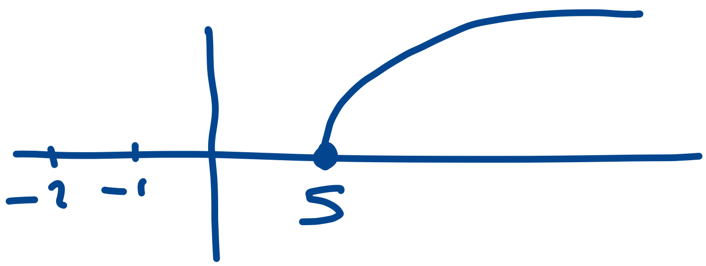
Example: Find the interval(s) over which the function $f(x) = x^2 + 2$ is continuous.

Answer: $(-\infty, \infty)$

Example: Find the interval(s) over which the function $f(x) = \sqrt{x-5}$ is continuous.

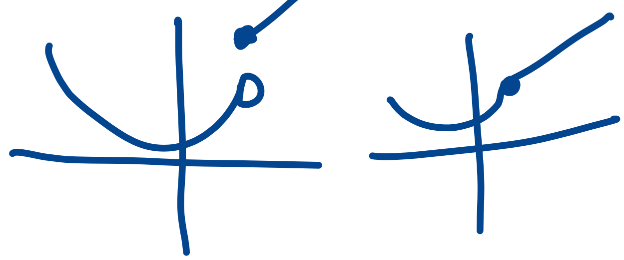
Answer: $[5, \infty)$

included



Example: Find the interval(s) over which the function $f(x) = \dfrac{1}{x-2}$ is continuous.

Answer: $(-\infty, 2) \cup (2, \infty)$



How to work with piece-wise functions:

Identify possible points of discontinuity (breaks, vertical asymptotes, etc.)

Check each using the 3 steps (function value, RHL, LHL)

Recall: For f to be continuous at c, we need: $f(c) = \lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x)$

Example: Find all points of discontinuity and classify them:

$$f(x) = \begin{cases} x^2+6, & \text{if } x<1 \\ 5x, & \text{if } 1 \le x \end{cases}$$

Points to investigate: $x = 1$

$x =$ 1 Compare the two-sided limits and function value:

Function Value: $f(1) = 5\cdot 1 = 5$

RHL: $\lim_{x\to 1^+} (5x) = 5\cdot 1 = 5$ $\neq$

LHL: $\lim_{x\to 1^-} (x^2+6) = 1^2+6 = 7$

$\lim_{x\to 1} f(x)$: DNE

Conclusion: The function is discontinuous at $x=1$.

(jump discty.)

RHL $\neq$ LHL

similar to WH 2

Example: Find all points of discontinuity and classify them:

$$f(x)=\begin{cases} x-1, & \text{if } x<4 \\ \dfrac{12}{x}, & \text{if } 4\le x<6 \\ \dfrac{10}{x-8}, & \text{if } x>6 \end{cases}$$

Points to investigate: $x=4$, $x=6$, ~~$x\neq 0$~~ ?, $x=8$

$x =$ 4 Compare the two-sided limits and function value:

Function Value: $f(4)=12/4=3$

RHL: $12/4=3$

LHL: $4-1=3$

$= \checkmark \quad \lim_{x\to 4} f(x)=3 \quad = \checkmark$

Conclusion: continuous at $x=4$

$x =$ 6

Function Value: $f(6)$: undefined

RHL: $10/(6-8) = -5$

LHL: $12/6 = 2$

$\neq \quad \lim_{x\to 6} f(x)$: DNE

Conclusion: discts at $x=6$ (jump discontinuity)

$x =$ 8

Function Value:

RHL:

LHL:

Conclusion:

$x=8$ is a V.A. for $f(x)$

infinite discontinuity at $x=8$.

discts: discontinuous

discty: discontinuity

WHW 2

$$\lim_{x\to 2^-}(Ax^2-14) = 4A-14$$

Example: Find the values of A and B so that the function is continuous everywhere.

quad → $$f(x)=\begin{cases} Ax^2-14, & \text{if } x<2 \\ 10, & \text{if } x=2 \\ Bx, & \text{if } x>2 \end{cases}$$ ← line

$f(2)=10$

$$\lim_{x\to 2^+}(Bx) = B\cdot 2 = 2B$$

Need: $f(x)$ is continuous at $x=2$.

Need: $$f(2) = \underbrace{\lim_{x\to 2} f(x)}$$

$$f(2) = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$$

$$10 = 2B = 4A-14$$

$\boxed{B=5}$

$4A-14=10$

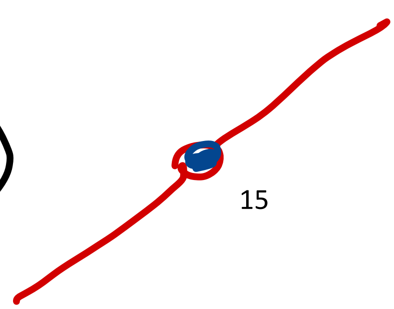
$4A=24 \rightarrow \boxed{A=6}$

Example: The function $f(x)=\dfrac{x^2-1}{x-1}$ is continuous everywhere except for $x=1$.

Redefine this function so that it is continuous everywhere.

$$f(x)=\begin{cases} \dfrac{x^2-1}{x-1}, & \text{if } x\neq 1 \\ 2, & \text{if } x=1 \end{cases}$$

$$f(x)=\frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}}=(x+1)$$



H/W

Exercise: Find the value of A so that the function is continuous everywhere.

$$f(x)=\begin{cases} Ax^2, & \text{if } 5<x \\ 2x+A, & \text{if } x\geq 5 \end{cases}$$

Exercise: Find all points of discontinuity and classify them:

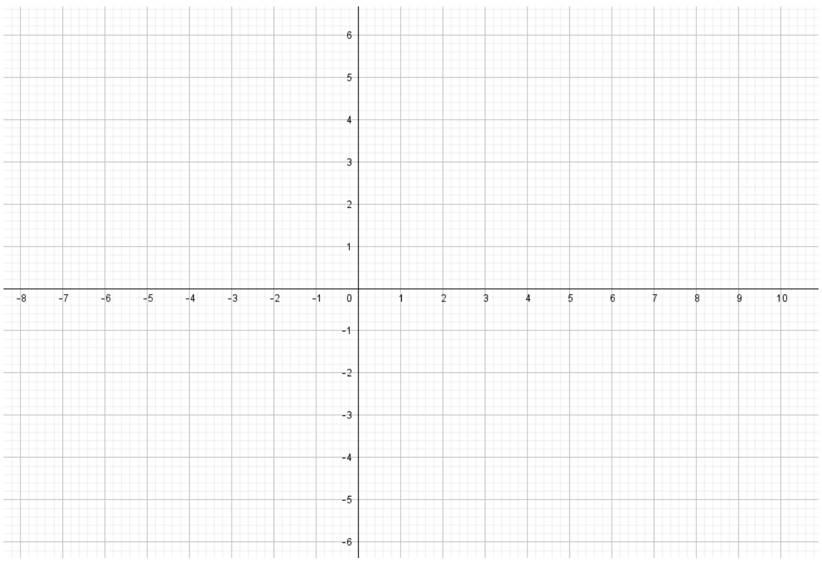
$$f(x)=\begin{cases} \dfrac{1}{x}, & \text{if } x<1 \\ \sqrt{x}, & \text{if } 1<x<4 \\ x-2, & \text{if } 4<x\leq 5 \\ \dfrac{12}{x-1}, & \text{if } 5<x \end{cases}$$

Exercise: Find all points of discontinuity and classify them:

$$f(x)=\begin{cases} 2x, & \text{if } x<0 \\ \sqrt{x}, & \text{if } 0\leq x<1 \\ 2x-1, & \text{if } 1<x\leq 4 \\ \dfrac{2}{x-10}, & \text{if } 4<x \end{cases}$$

Exercise: Graph a function satisfying all of the properties below:

- The function has removable discontinuity at $x=-1$.
- The function has jump discontinuity at $x=2$.
- The Function has infinity discontinuity at $x=6$.
- $\lim\limits_{x\to\infty} f(x)=0$
- $\lim\limits_{x\to-\infty} f(x)=-1$



1.4 is over