Completed notes

Math 2413- Calculus I

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- Check CASA calendar for due dates.
- Bring "blank notes" to class. Completed notes will be posted after class.
- Do your best to attend every lecture and lab. This is a 4 credit course because of the lab component.
- Study after every lecture; work on the quiz covering the topic we cover on the lecture immediately afterwards. Retake your quizzes for more practice.
- Get help when you need help; bring your questions to the labs, or my office hours. We also have tutoring options on campus.
- Make sure you are a member of our team; check the discussion channel for announcements. You can post questions there. Make sure MS teams notifications are ON so that you are notifies when we make announcements there.
- Respect your friends in class; stay away from distractive behavior. Do your best to concentrate on the lecture. Be considerate of others.
- If you email me, mention the course code in the subject line.

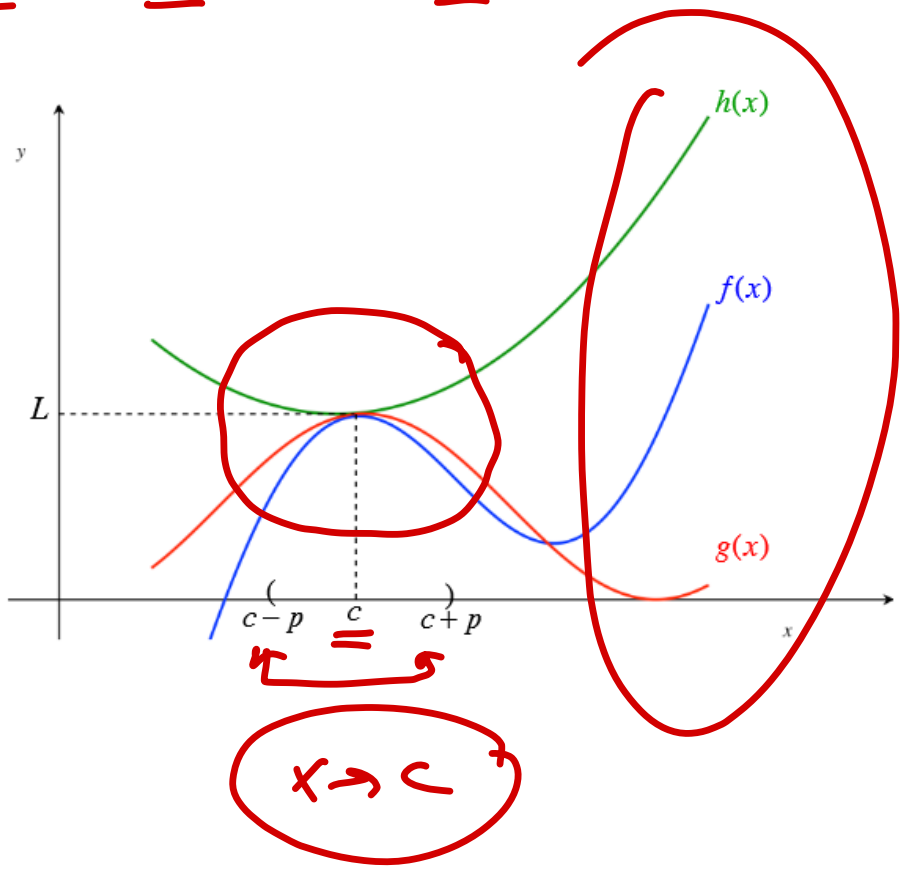
Section 1.6 – The Pinching Theorem

The Sandwich Theorem

Theorem: The Pinching Theorem

Let $p>0$ and c be a real number. Suppose that $f(x)\le g(x)\le h(x)$ for any x in the interval $(c-p,c+p)$ (except possibly at $x=c$).

If $\lim_{x\to c} f(x)=\lim_{x\to c} h(x)=L$, then $\lim_{x\to c} g(x)=L$.



$x\to c$

Trigonometric Limits:

$\lim_{x\to 0}\sin(x)=$ $\sin(0)=0$

$\lim_{x\to 0}\cos(x)=$ $\cos(0)=1$

In general, $\lim_{x\to c}\sin(x)=\sin(c)$ and $\lim_{x\to c}\cos(c)=\cos(c)$.

That is, step 1 is direct substitution as before.

Example:

$\lim_{x\to\pi}\left(\frac{1+2\cos(x)}{4x}\right)=?$ $\frac{1+2\cos(\pi)}{4\cdot\pi}=\frac{1-2}{4\pi}=\frac{-1}{4\pi}$ ✓

Example:

$\lim_{x\to\frac{\pi}{2}}\left(\frac{4+\sin(x)}{2+\cos(x)}\right)=?$ $\frac{4+\sin\left(\frac{\pi}{2}\right)}{2+\cos\left(\frac{\pi}{2}\right)}=\frac{4+1}{2+0}=\frac{5}{2}$

ex $\lim_{x\to 0}\frac{\sin(x)}{x}=\frac{\sin(0)}{0}=\frac{0}{0}$?

Now, we look at two very important limits.

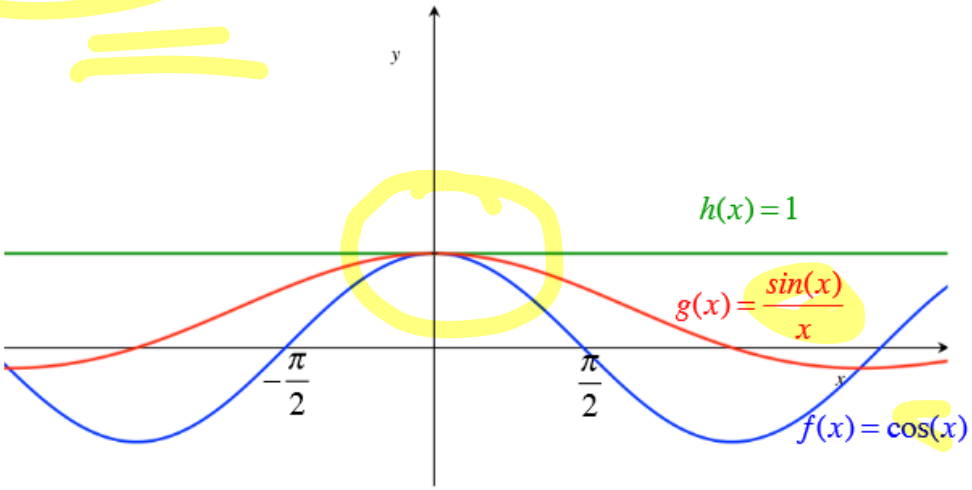
$$\lim_{x\to 0}\frac{\sin(x)}{x}=?\quad\text{and}\quad\lim_{x\to 0}\frac{1-\cos(x)}{x}=?.$$

If you try to use direct substitution on these limits, you'll get the indeterminate form 0/0.

Idea: For the function $g(x)=\dfrac{sin(x)}{x}$, if $-\dfrac{\pi}{2}\le x\le\dfrac{\pi}{2}$, then $cos(x)\le\dfrac{sin(x)}{x}\le 1$.

Since $\lim_{x\to 0} cos(x)=1$ and $\lim_{x\to 0} 1=1$, by the Pinching theorem, we conclude that

$$\lim_{x\to 0}\frac{sin(x)}{x}=1.$$



ex: $\lim_{x\to 0}\dfrac{1-\cos(x)}{x}=\dfrac{1-1}{0}=\dfrac{0}{0}$?

Fact: $\lim_{x\to 0}\frac{\sin(x)}{x}=1$ and $\lim_{x\to 0}\frac{1-\cos(x)}{x}=0$.

Fact: $\lim_{x\to 0}\frac{\sin(ax)}{bx}=\frac{a}{b}$ and $\lim_{x\to 0}\frac{1-\cos(ax)}{bx}=0$.

ALWAYS pay attention to what value x is approaching. These facts only apply in certain cases (0/0).

Examples:

a) $\lim_{x\to 0}\frac{\sin(5x)}{x}=$ by fact $\frac{5}{1}=5$

$\frac{\sin(0)}{0}=\frac{0}{0}$

b) $\lim_{x\to 0}\frac{\sin(2x)}{6x}=$ $\frac{2}{6}=\frac{1}{3}$

$\frac{0}{0}$

c) $\lim_{x\to 0}\frac{2x}{\sin(7x)}=$ by fact $\frac{2}{7}$

$\frac{0}{0}=$

FACT

$$\lim_{x\to 0}\frac{1-\cos(x)}{x}=0.$$

$$\lim_{x\to 0}\frac{1-\cos(ax)}{bx}=0.$$

Examples:

a) $\lim_{x\to 0}\frac{1-\cos(2x)}{5x}=$ 0

$\frac{0}{0}$

b) $\lim_{x\to 0}\frac{1-\cos(4x)}{9x}=$ 0

$\frac{0}{0}$

Examples:

a) $\lim_{x\to 0}\frac{\sin^2(4x)}{5x}=$

$\frac{0^2}{0}=\frac{0}{0}$?

$f\cdot g$

$$\lim_{x\to 0}\left[\frac{\sin(4x)}{5x}\cdot\sin(4x)\right]$$

$$=\lim_{x\to 0}\left[\frac{\sin(4x)}{5x}\right]\cdot\lim_{x\to 0}\left[\sin(4x)\right]$$

(by fact) (sub.)

$$=\frac{4}{5}\cdot 0$$

$$=0$$

b) $\lim_{x\to 0}\frac{\sin(4x)}{\sin(9x)}=$

$$\frac{A}{B} = \frac{A}{x} \cdot \frac{x}{B}$$

$$\lim_{x \to 0} \frac{\sin(4x)}{\sin(9x)}$$

$$= \lim_{x \to 0} \left[\underbrace{\frac{\sin(4x)}{x}}_{\downarrow \text{by fact}} \cdot \underbrace{\frac{x}{\sin(9x)}}_{\downarrow \text{by fact}} \right]$$

$$= \frac{4}{1} \cdot \frac{1}{9}$$

$$= \boxed{\frac{4}{9}}$$

save as: $\lim_{x\to 0} \frac{\sin(x^2)}{5\cdot x^2} = \lim_{t\to 0} \frac{\sin(t)}{5t}$ ($x^2 \to 0$, $t = x^2$)

Examples:

a) $\lim_{h\to 0} \frac{\sin(h^2)}{5h^2} =$ (by fact) $\boxed{\frac{1}{5}}$

ex: $\lim_{x\to 0} \frac{\sin(4x^3)}{5x^3} = \frac{4}{5}$

b) $\lim_{x\to 0} \frac{5x^2}{\sin^2(6x)} =$ ($5x\cdot x$)

$= \lim_{x\to 0} \left[\frac{5x}{\sin(6x)} \cdot \frac{x}{\sin(6x)}\right]$

(by fact) (by fact)

$= \frac{5}{6} \cdot \frac{1}{6} = \boxed{\frac{5}{36}}$

c) $\lim_{t\to 0} \frac{\sin(5t^2)}{t^2\cos(2t)} =$

$\lim_{t\to 0} \left[\frac{1}{\cos(2t)}\right]$

0/0

$= \lim_{t\to 0} \frac{\sin(5t^2)}{t^2} \cdot \frac{1}{\cos(2t)}$

(by fact) (direct substitution)

$\frac{5}{1} \cdot \frac{1}{1}$

$= \boxed{5}$

$\tan(0)=0$

Examples:

a) $\lim_{x\to 0}\frac{\tan(5x)}{4x}=$

$\lim_{x\to 0}\frac{\sin(5x)}{\cos(5x)\cdot 4x}$

$\frac{0}{0}$

$=\lim_{x\to 0}\frac{\sin(5x)}{4x}\cdot\frac{1}{\cos(5x)}$

$\frac{\sin(5x)}{4x}$ ↙ fact

$\frac{1}{\cos(5x)}$ ↙ subst.

$=\frac{5}{4}\cdot\frac{1}{1}=\boxed{\frac{5}{4}}$

b) $\lim_{x\to 0}\frac{\tan(6x)}{\tan(4x)}=$

$=\lim_{x\to 0}\frac{\tan(6x)}{x}\cdot\frac{x}{\tan(4x)}$

$=\frac{6}{1}\cdot\frac{1}{4}=\frac{6}{4}=\boxed{\frac{3}{2}}$

Sometimes, trig identities might be needed to simplify the given expression.

Examples:

a) $\lim_{x\to 0} \frac{1-\cos^2(2x)}{7x} =$ identity

$= \lim_{x\to 0} \frac{\sin^2(2x)}{7x}$

$\frac{0}{0}$

$= \lim_{x\to 0} \frac{\sin(2x)}{7x} \cdot \sin(2x) = \frac{2}{7} \cdot 0$

$= \boxed{0}$

b) $\lim_{x\to 0} x^2 \csc\left(5x^2\right) =$

$= \lim_{x\to 0} \frac{x^2}{\sin(5x^2)} = \boxed{\frac{1}{5}}$

c) $\lim_{x\to 0} \frac{\cos(x)\tan(x)}{x} =$

$\lim_{x\to 0} \frac{\sin(x)}{x} = \boxed{1}$ ← fact

$\frac{0}{0}$

$\cos(x) \cdot \frac{\sin(x)}{\cos(x)} = \sin(x)$

d) $\lim_{x\to 0} \frac{1-\sec^2(4x)}{x^2} =$

$= \lim_{x\to 0} \frac{\tan^2(4x)}{x^2}$

$= \lim_{x\to 0} \frac{\tan(4x)}{x} \cdot \frac{\tan(4x)}{x}$

$= \frac{4}{1} \cdot \frac{4}{1} = \boxed{16}$

ALWAYS pay attention to what value the variable is approaching; plug in first!

Review identities and unit circle from Precalculus.

Example:

$$\lim_{x\to\pi/4}\frac{\tan(3x)}{8x} = \frac{\tan(3\pi/4)}{8\cdot\frac{\pi}{4}} = \frac{-1}{2\pi}$$

1.6 is over

Exercise: $\lim_{x\to\pi}\frac{sin(x)}{x-\pi} =$ $\lim_{h\to 0}\frac{\sin(h+\pi)}{h} = \lim_{h\to 0}\frac{-\sin(h)}{h}$

$\frac{0}{0}$ let $h = x-\pi$

$\Rightarrow x = h+\pi$

$= \boxed{-1}$

Note: $\sin(h+\pi) = -\sin(h)$ [fact from Precalc]

NEXT: Chapter 2.