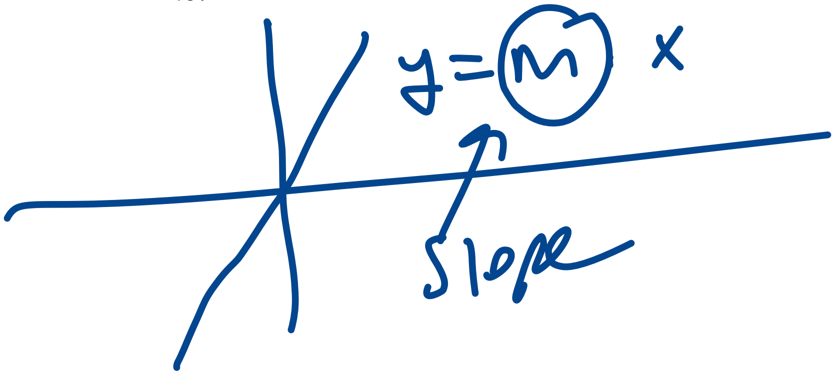
Completed

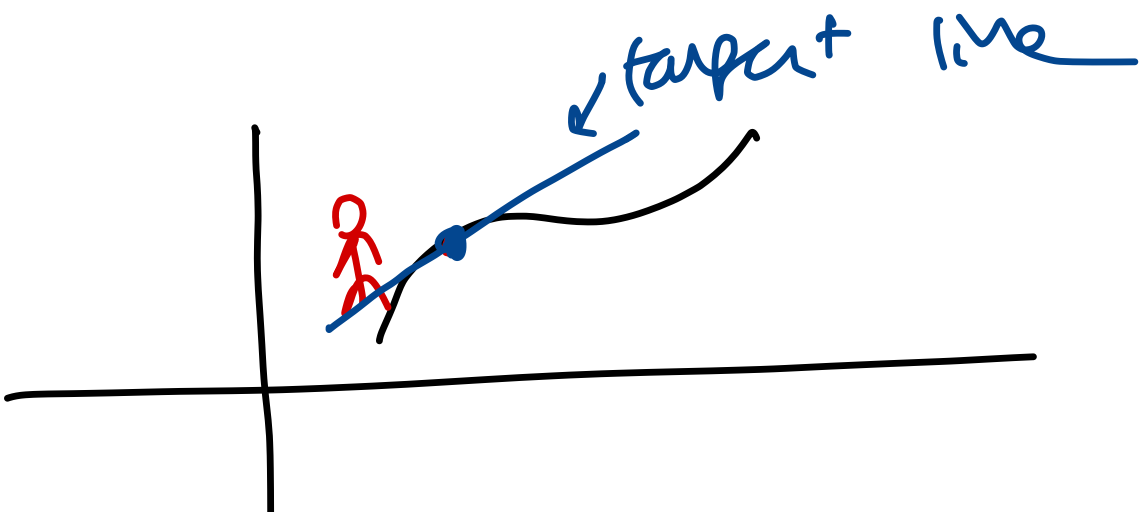
Math 2413- Calculus I

Dr. Melahat Almus

Email: malmus@uh.edu

- Check CASA calendar for due dates.
- Bring “blank notes” to class. Completed notes will be posted after class.
- Do your best to attend every lecture and lab. This is a 4-credit course because of the lab component.
- Study after every lecture; work on the quiz covering the topic we cover on the lecture immediately afterwards. Retake your quizzes for more practice.
- Get help when you need help; bring your questions to the labs, or my office hours. We also have tutoring options on campus.
- Respect your friends in class; stay away from distractive behavior. Do your best to concentrate on the lecture.
- If you email me, mention the course code in the subject line. Email is the best way to communicate with me outside of class. Teams chat messages are not monitored or replied to.

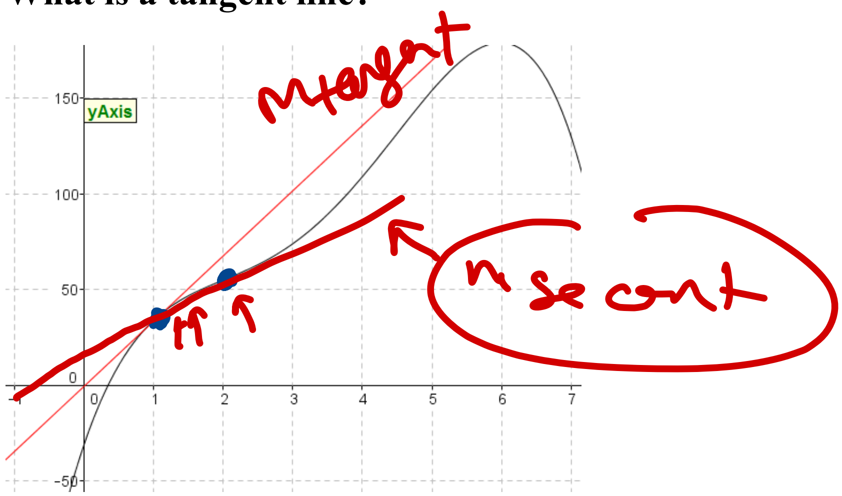




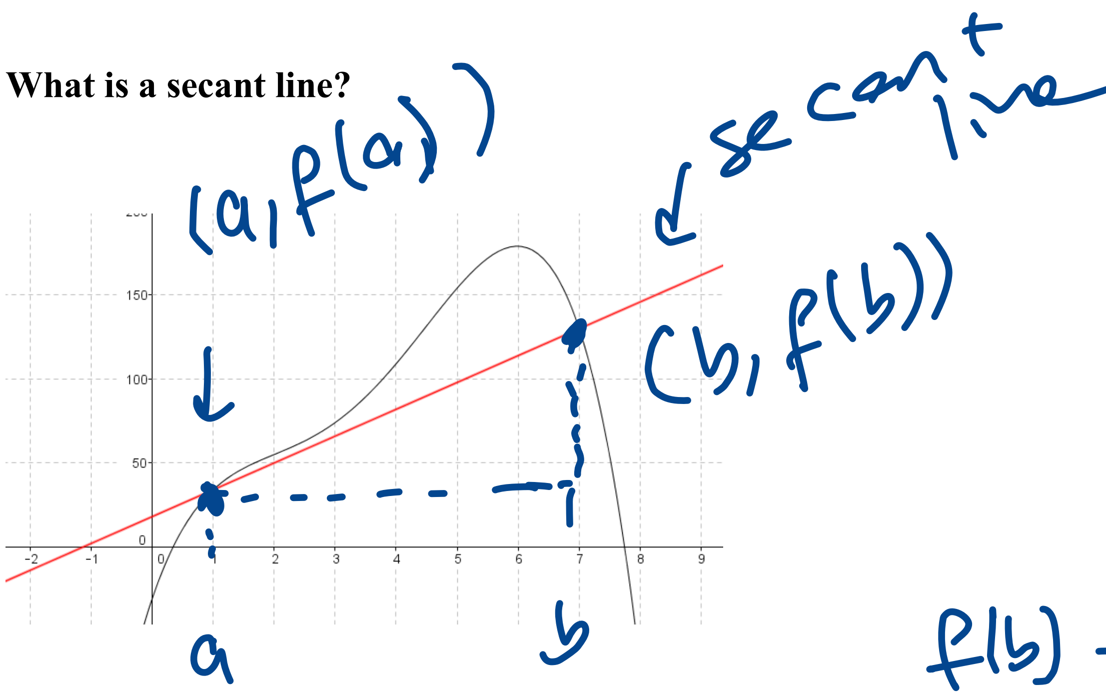
Chapter 2

Section 2.1 – The Definition of the Derivative

What is a tangent line?

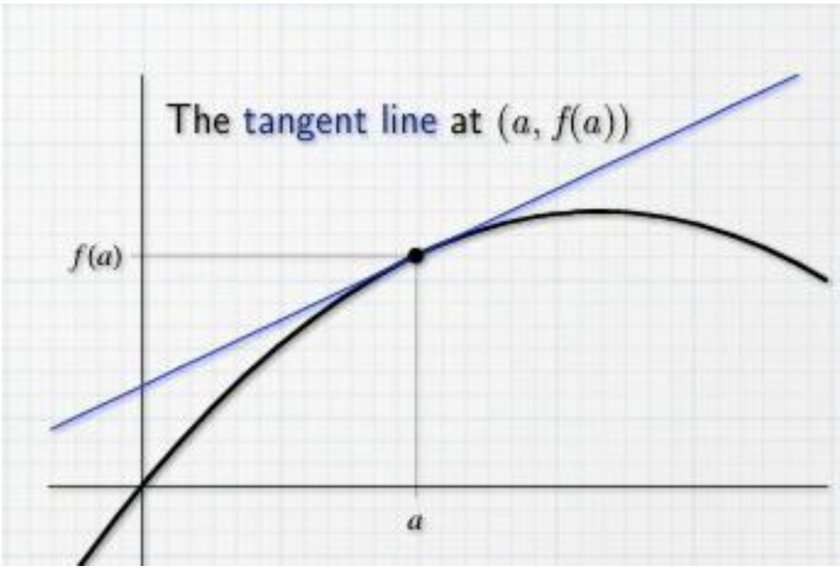


What is a secant line?

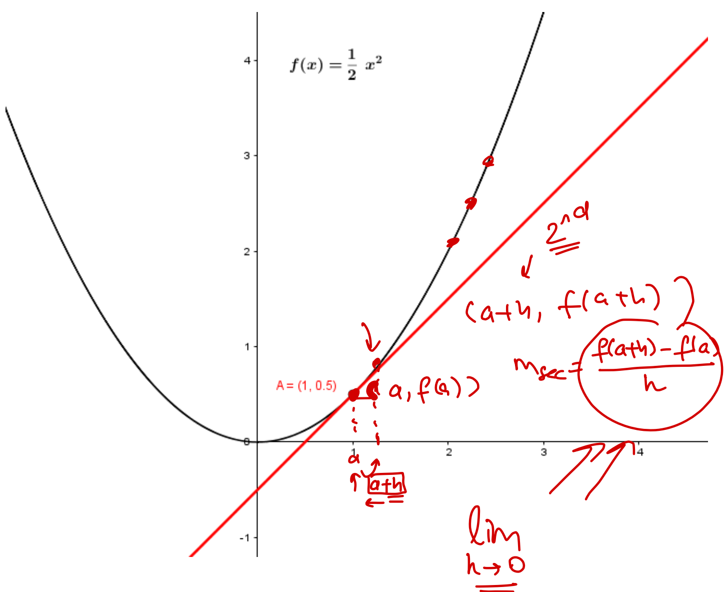


$$m_{secant} = \frac{f(b) - f(a)}{b - a}$$

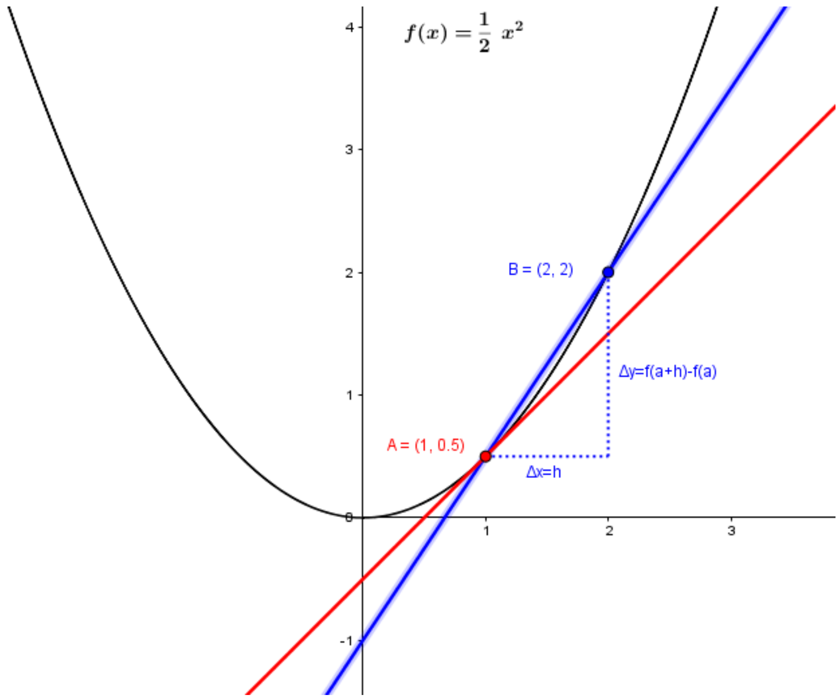
Goal: Given a function, find the slope of the tangent line at the point $(a, f(a))$.



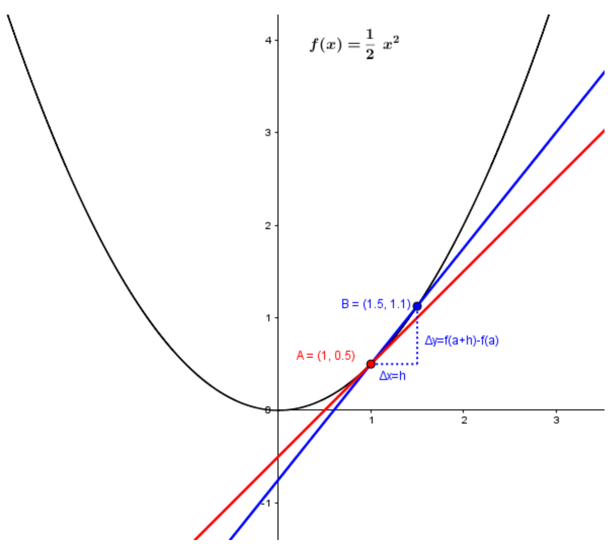
Let's try to "approximate" the slope of the tangent line at the point $(a, f(a))$.



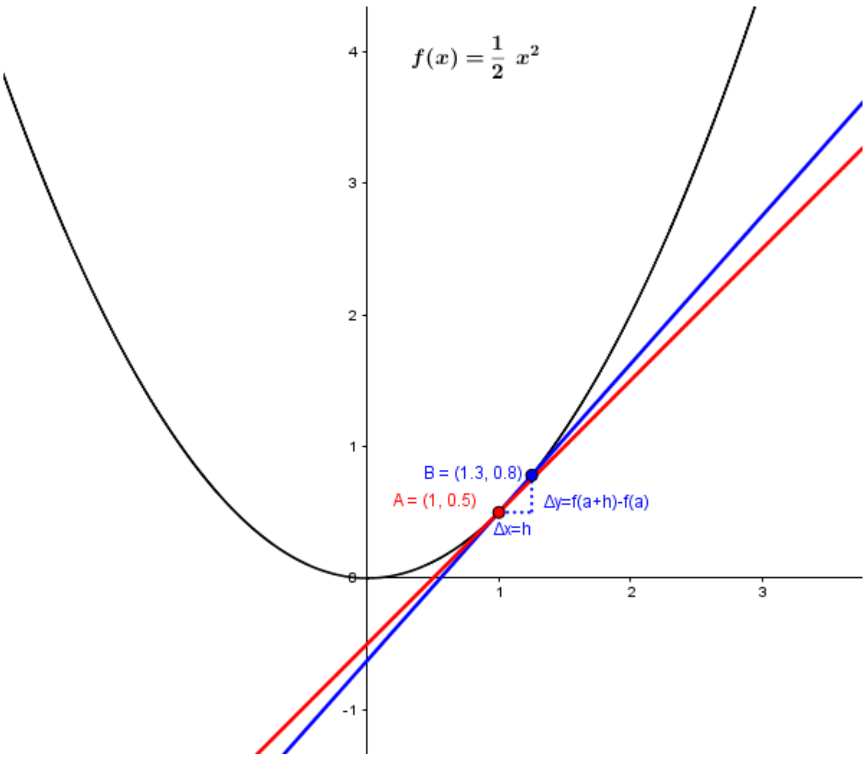
Pick another point $P\left(a+h, f\left(a+h\right)\right)$ on the curve and draw the secant line:



The slope of this secant line is: $\dfrac{f\left(a+h\right)-f\left(a\right)}{a+h-a}=\dfrac{f\left(a+h\right)-f\left(a\right)}{h}$. This expression is called the *difference quotient* of f at $\left(a, f\left(a\right)\right)$.



To make a better approximation, we need to move the two points closer together. In other words, we need to make h smaller.



Now, the slope of this new secant line is a better approximation for the slope of the tangent line. We want to let the distance between two points get smaller and smaller in order to minimize the error. That is, **we need to let h approach 0 by taking limits.** This gives us a formula for finding the slope of the tangent line.

The slope of a tangent line to a function $f(x)$ at the point $(a, f(a))$ is

$$\boxed{\lim_{h \to 0} \frac{f(a+h)-f(a)}{h},}$$

if this limit exists.

$\lim_{h \to 0}$ [difference quotient]

Definition: A function f is said to be **differentiable at** a, if the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \text{ exists.}$$

The value of this limit is called the **derivative of** f **at** a and is denoted by $f'(a)$ (read as "f prime at a" or "f prime of a").

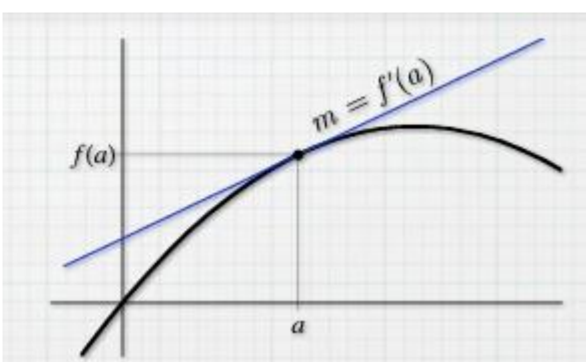
$f'(a) : m_{tangent}$

In other words, we define

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided that the limit exists.

Note that $f'(a)$ is a real number and gives the slope of the tangent line at $(a, f(a))$. This number is also said to be the **slope of the graph of** f **at** $(a, f(a))$.



$f(a)$ a

Remarks: 1. Since $f'(a)$ is the slope of the tangent line at $(a, f(a))$, using the point-slope equation of a line, we can write the equation of the tangent line:

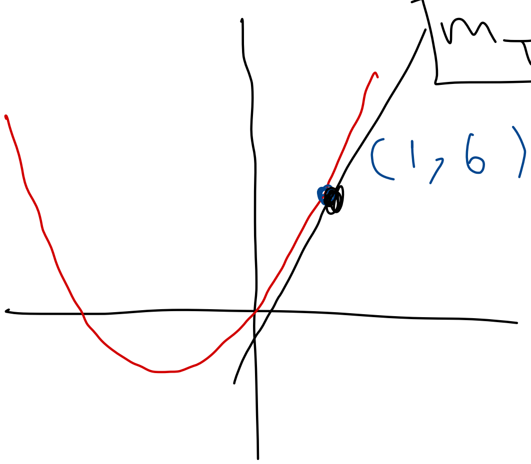
$$\rightarrow \quad y - f(a) = f'(a)(x-a).$$

$$y - y_0 = m_{tang}(x - x_0)$$

2. $f'(a)$ is often referred to as the **rate of change** in $f(x)$ at $x = a$.

Example 1: Find the slope of the line tangent to the function $f(x) = x^2 + 5x$ at the point $(1,6)$. Find the equation of the tangent line.

$\boxed{m_T = 7}$



$$f'(1) = \lim_{h \to 0} \left(\frac{f(1+h) - f(1)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{(1+h)^2 + 5(1+h) - 6}{h} \right) = \frac{0}{0}$$

$$f'(1) = \lim_{h \to 0} \left(\frac{1 + 2h + h^2 + 5 + 5h - 6}{h} \right)$$

$$f'(1) = \lim_{h \to 0} \left(\frac{h^2 + 7h}{h} \right) = \lim_{h \to 0} (h+7) = 0 + 7 = \boxed{7}$$

$$\frac{\cancel{h}(h+7)}{\cancel{h}}$$

The Derivative as a Function

Definition:

Given a function f, the **derivative of** f is the function f' defined as:

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$, provided the limit exists.

The domain of $f'(x)$ is the set of all points where the defining limit exists, that is, all x for which f is **differentiable**.

To **differentiate** a function means to find its derivative.

Remark: 1. To apply this definition, f must be defined at some open interval containing x. It is important to also note that when taking the limit, the variable is h and x is fixed.

2. Wherever $f'(x)$ is defined, it is the slope of the graph of f at $(x, f(x))$ and it also gives the rate of change in $f(x)$.

$f'(x)$: the derivative of $f(x)$

Let's revisit Example 1: Given $f(x)=x^2+5x$, find $f'(x)$. And then, find the slope of tangent line at (1,6).

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\left(\frac{(x+h)^2+5(x+h)-(x^2+5x)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{\cancel{x^2}+2xh+h^2+\cancel{5x}+5h\cancel{-x^2}\cancel{-5x}}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{2xh+h^2+5h}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{(2x+h+5)\cdot\cancel{h}}{\cancel{h}}\right)$$

$$=\lim_{h\to 0}(2x+h+5)=\boxed{2x+0+5}$$

$$\boxed{f'(x)=2x+5}$$

$$f'(1)=2\cdot 1+5=7$$

$$f'(2)=2\cdot 2+5=9$$

$$(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b}) = a-b$$

Example 2: Given $f(x)=\sqrt{x+2}$, find $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(\sqrt{x+h+2}-\sqrt{x+2})\cdot(\sqrt{x+h+2}+\sqrt{x+2})}{h\cdot(\sqrt{x+h+2}+\sqrt{x+2})}$$

$$= \lim_{h\to 0} \frac{(x+h+2)-(x+2)}{h\cdot(\sqrt{x+h+2}+\sqrt{x+2})}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}\cdot(\sqrt{x+h+2}+\sqrt{x+2})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h+2}+\sqrt{x+2}}$$

$$= \frac{1}{\sqrt{x+0+2}+\sqrt{x+2}} = \frac{1}{2\sqrt{x+2}}$$

$f(x) = \sqrt{x+2}$ ←

$\Rightarrow \quad f'(x) = \dfrac{1}{2\sqrt{x+2}}$

m tangent line at $x=0$

$f'(0) = \dfrac{1}{2\sqrt{2}}$

Equation of the tangent line at $x=0$.

$m_t = \dfrac{1}{2\sqrt{2}} = \dfrac{\sqrt{2}}{4}$

$y - f(0) = \dfrac{\sqrt{2}}{4}(x-0)$

$y - \sqrt{2} = \dfrac{\sqrt{2}}{4} \cdot x$

$$\frac{1}{x+h+5}\;-\;\frac{1}{x+5}$$

$$\frac{x+h+5}{(x+5)} \qquad \frac{x+5}{(x+h+5)}$$

rational

Example 3: Use the limit definition to find the derivative of $f(x)=\dfrac{1}{x+5}$.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{\dfrac{1}{x+h+5}-\dfrac{1}{x+5}}{h}$$

$$=\lim_{h\to 0}\frac{\dfrac{(x+5)-(x+h+5)}{(x+h+5)\cdot(x+5)}}{h}$$

$$=\lim_{h\to 0}\frac{\dfrac{-h}{(x+h+5)\cdot(x+5)}}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{-h}^{\,(-1)}}{(x+h+5)(x+5)\cdot\cancel{h}}$$

$$=\lim_{h\to 0}\frac{-1}{(x+h+5)(x+5)}$$

$$= \frac{-1}{(x+0+5)\cdot(x+5)}$$

$$= \boxed{\frac{-1}{(x+5)^2}}$$

$$f(x) = \frac{1}{x+5} \Rightarrow \boxed{f'(x) = \frac{-1}{(x+5)^2}}$$

ex: $f(x) = \frac{1}{x+5}$; slope of the tangent line at $x=1$

$$f'(1) = \frac{-1}{(1+5)^2} = -\frac{1}{36}$$

$$\frac{1}{7} - \frac{1}{9}$$

$$(9) \qquad (7)$$

$$\frac{\frac{A}{B}}{\frac{C}{1}} = \frac{A}{B} \cdot \frac{1}{C}$$

$$= \frac{A}{B \cdot C}$$

$$\frac{\frac{A}{5}}{7} = \frac{A}{5 \cdot 7}$$

$$\frac{f(x+h) - f(x)}{h}$$

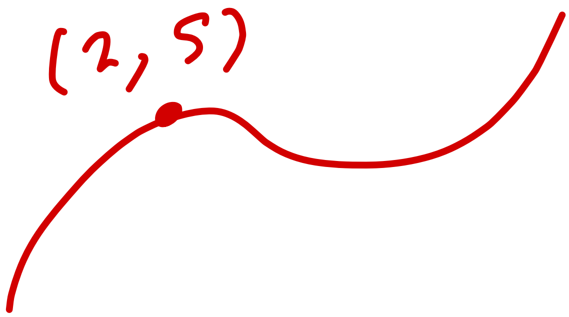
Example 4: Given $f'(c) = \lim_{h \to 0} \frac{\sqrt{5+h} - \sqrt{5}}{h}$, identify the function $f(x)$ and the number c.

$c = 5$, $\quad f(x) = \sqrt{x}$

Example 5: Given $f(2) = 5$ and $f'(2) = 4$, find the equation of the tangent line to $f(x)$ at $x = 2$.

$m_{tangent} = 4$

$(2, 5)$



$y - 5 = 4(x - 2)$

Differentiability

A function f is **differentiable at** a if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \text{ exists.}$$

If this limit fails to exist, we say that the function is not differentiable at a.

If a function is differentiable at every number in an open interval I, we say that the function is **differentiable on** I. For example, the function $f(x) = \sqrt{x}$ is differentiable on the interval $(0, \infty)$.

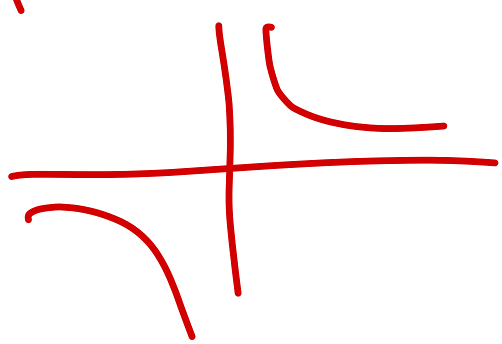
Not cts at $x=a \Rightarrow$ Not diff. at $\underline{\underline{x=a}}$.

What is the relation between continuity and differentiability?

Fact: If f is differentiable at a, then it is continuous at a.

However, the converse of this statement is not always true. A function can be continuous at a without being differentiable. For example, $f(x) = |x|$ is continuous at every real number but it is not differentiable at 0.

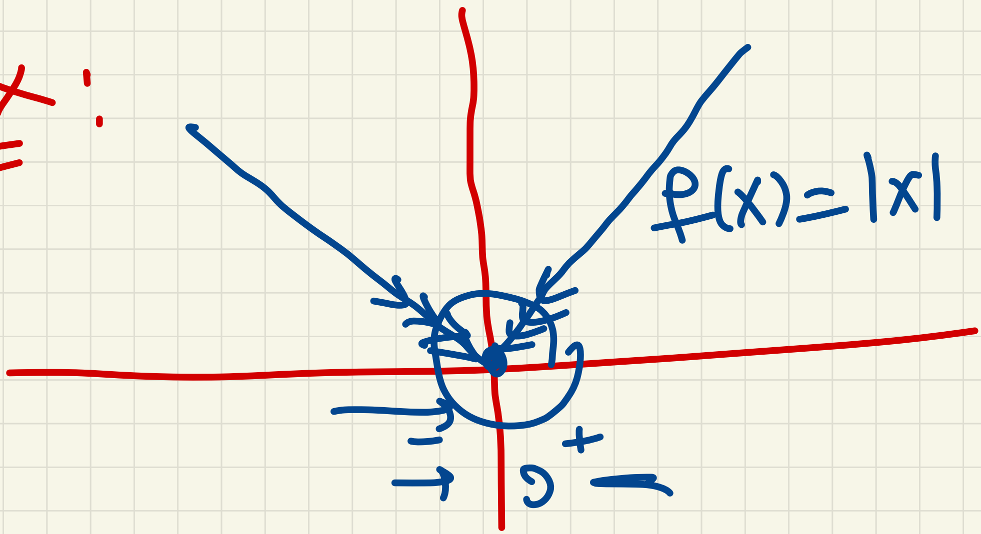
$f(x) = \frac{1}{x}$ is not $\underline{\underline{\text{cts}}}$ at $x=0$.



f is cts at $x=0$

$\overset{?}{\Longrightarrow}$ f is differentiable at $x=0$?

<ins>ex</ins>:

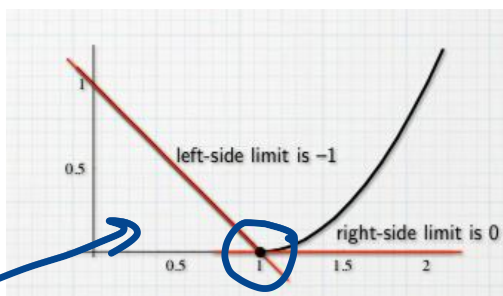


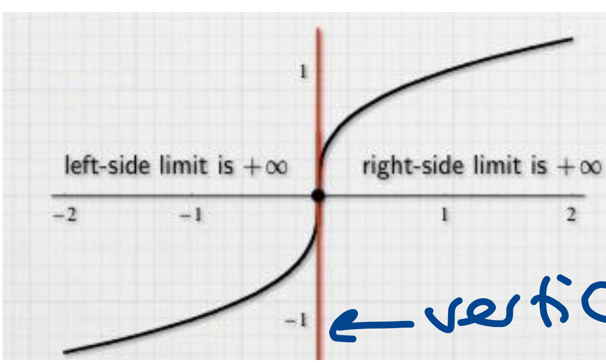
When is a function not differentiable at a point?

- The first problem that makes a function f not differentiable at a is discontinuity at a.

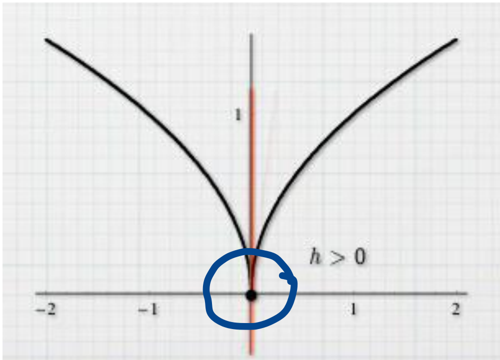
Example: $f(x) = \begin{cases} 2x, x<0 \\ x^2+1, x \geq 0 \end{cases}$ is not differentiable at $x=0$ because it is not continuous at $x=0$.

- Another issue is having a "sharp corner", a cusp, or a vertical tangent.





← vertical tangent line



The first graph is not differentiable at 1 since one-sided derivatives do not coincide. The left-hand derivative at 1 is -1, but the right-hand derivative is 0. We may refer to this case as f having a *sharp corner* at 1. On the second graph, there is a *vertical tangent* at 0. And, on the last graph there is a *cusp* at 0. We will discuss vertical tangents and cusps in detail later.

Remark: Polynomials are differentiable on $(-\infty,\infty)$. All polynomials have continuous graphs, with no sharp corners or cusps.

Example: $f(x)=x^3+2x-4$ is a polynomial; so it is differentiable everywhere.

Example 6: Is this function differentiable?

$$f(x)=\begin{cases}5x, & \text{if } x>1\\ x^2+4, & \text{if } x\le 1\end{cases}$$

Since you don't know the rules of the derivative yet - Hint: the derivative of "5x" is 5 and the derivative of x^2+4 is $2x$.

Step 1: Check whether the function is continuous or not at x=1.

Step 2: Check whether right and left hand slopes match:

Derivative from the left side: $f'(1^-)=5\cdot 1=5$

Derivative from the right side: $f'(1^+)=2\cdot 1^2=2$

More on this later.