Math 2413- Calculus I

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- Check CASA calendar for due dates.
- Bring "blank notes" to class. Completed notes will be posted after class.
- Do your best to attend every lecture and lab.
- Study after every lecture; work on the quiz covering the topic we cover on the lecture immediately afterwards. Retake your quizzes for more practice.
- Get help when you need help; bring your questions to the labs, or my office hours. We also have tutoring options on campus.
- Respect your friends in class; stay away from distractive behavior. Do your best to concentrate on the lecture.
- If you email me, mention the course code in the subject line. Email is the best way to communicate with me outside of class. Teams chat messages are not monitored or replied to.

Today: S 2.3

Online Quiz 7: S 2.2

Online Quiz 8: S 2.3

Take & retake!

WHW 4: 2.3 & 2.4 (next Tuesday)

Lab quiz this week: 2.1 & 2.2

Lab quiz next week: 2.3 & 2.4

Warm up questions covering the previous section. Answer them before class.

Question# If $f(x)=2x^3+5x^2-2x$, $f'(1)=?$

a) 5
b) 14
c) 12
d) 8
e) None

Question# If $f(x)=\dfrac{5x^3+x^2+x}{x^2}$, $f'(1)=?$

a) 7
b) 5
c) 4
d) 2
e) None

Question# If $f(x)=5\cos(x)+4\sin(x)$, $f'(\pi)=?$

a) -5
b) 4
c) -4
d) 1
e) None

Question# If $f(x)=\tan(x)+6\cot(x)$, $f'\left(\dfrac{\pi}{4}\right)=?$

a) -10
b) 7
c) -12
d) 14
e) None

Recall

Slope of tangent line at $x=c$

Step 1: Find $f'(x)$

Step 2: Plug in $m_{tan} = f'(c)$

Slope of normal line at $x=c$

Step 1: find $f'(x)$

Step 2: Plug in: $m_{tan} = f'(c)$

Step 3: negative reciprocal: $m_N = \frac{-1}{m_{tan}}$

DAY 7

Q# 1 $f(x) = x^3 - 3x + 1$

What is the slope of the tangent line at $x = 0$?

a) 1 b) 3 c) -3 d) 2 e) None

$f'(x) = 3x^2 - 3$

Q# 2 $f(x) = x^3 - 3x + 1$ $f'(2) = \boxed{9}$

What is the slope of the normal line at $x = 2$?

a) 9 b) $-1/9$ c) $1/9$ d) $1/2$ e) None

~~Q #~~ $f(x) = x^3 - 3x + 1$

Where does this function have horizontal tangent lines?

a) $x = 1, x = -1$

b) $x = 1, x = 0$

c) $x = 2, x = -2$

d) None

$3x^2 - 3 = 0$

$3x^2 = 3$

$x^2 = 1$

$x = \pm 1$

$$f'(x) = \frac{1}{2\sqrt{x}} + \frac{-1}{x^2}$$

Q#1 3 $\quad f(x) = \sqrt{x} + \frac{1}{x}$

$f'(1) = ?$

a) 2   b) 1/2   c) −1/2   d) None

Q#

$y = x^2 + x$

$\left.\frac{dy}{dx}\right|_{x=1} = ?$

a) 3   b) 2   c) 1   d) None

$$h(x) = \frac{x+1}{x^2+4}$$

$$g(x) = x^2 \cdot \cos(x)$$

$f(x) = 2x \cdot \sin(x)$

Section 2.3 – Differentiation Rules

Theorem: The Product Rule

If f and g are differentiable at x, then so is the product fg. Moreover,

$$(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$$

This formula may be written as:

$$(uv)' = u'v + uv' \quad \text{or} \quad \frac{d}{dx}(u \cdot v) = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}.$$

This rule can be extended to the product of more functions:

$$(uvw)' = u'vw + uv'w + uvw' \quad \text{or}$$

$$\frac{d}{dx}(u \cdot v \cdot w) = \frac{du}{dx} \cdot v \cdot w + u \cdot \frac{dv}{dx} \cdot w + u \cdot v \cdot \frac{dw}{dx}.$$

$(u \cdot v \cdot w)' = u' \cdot v \cdot w + u \cdot v' \cdot w + u \cdot v \cdot w'$

Example: Find the derivative of $h(x)=x^3\cos(x)$.

Product rule

$h'(x) = (x^3)' \cdot \cos(x) + (x^3) \cdot (\cos(x))'$

$h'(x) = 3x^2 \cdot \cos(x) + x^3 \cdot -\sin(x)$

$h'(x) = 3x^2 \cos(x) - x^3 \cdot \sin(x)$

Example: If $y=(2x+5)(x^4+x^2)$, $\frac{dy}{dx}=?$

$\frac{dy}{dx} = \underbrace{\frac{d}{dx}(2x+5)} \cdot (x^4+x^2) + (2x+5) \cdot \underbrace{\frac{d(x^4+x^2)}{dx}}$

$\frac{dy}{dx} = 2 \cdot (x^4+x^2) + (2x+5) \cdot (4x^3+2x)$

$= 2x^4+2x^2 + \ldots$

Remark: Note that, to confirm our answer, we can compute the product first: $y=(2x+5)(x^4+x^2)=2x^5+2x^3+5x^4+5x^2$; and then take its derivative:

$$\frac{dy}{dx} \overset{\text{use rules about polynomials}}{=}$$

As you see, product rule gives us the same answer. It saves time in terms of not having to compute the product, especially if the expressions have more terms.

Imagine having to compute $(2x^2+5x+4)(x^4+2x^3+x^2-1)$ first to find the derivative. Instead, you can simply use the product rule.

$$\frac{d}{dx}\left[(2x^2+5x+4)(x^4+2x^3+x^2-1)\right]$$
$$=\frac{d}{dx}\left[(2x^2+5x+4)\right](x^4+2x^3+x^2-1)+(2x^2+5x+4)\frac{d}{dx}\left[(x^4+2x^3+x^2-1)\right]$$

Exercise: If $h(x)=x^2(1+\sin(x))(2+\cos(x))$, find $h'(x)=?$

Theorem: The Quotient Rule

If f and g are differentiable at x and $g(x) \neq 0$, then the quotient f/g is differentiable at x and

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{\left[g(x)\right]^2}$$

This formula may be written as:

$$\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2} .$$

a rational function.

Example: Find the derivative of $f(x) = \dfrac{7x}{x^2 + 5}$.

$$f'(x) = \frac{(7x)' \cdot (x^2+5) - (7x) \cdot (x^2+5)'}{(x^2+5)^2}$$

$$f'(x) = \frac{7 \cdot (x^2+5) - (7x)(2x)}{(x^2+5)^2}$$

$$f'(x) = \frac{7x^2 + 35 - 14x^2}{(x^2+5)^2} = \boxed{\frac{-7x^2+35}{(x^2+5)^2}}$$

$-7(x^2-5)$

or

$$= \frac{-7x^2+35}{x^4+10x^2+25}$$

$$\frac{x^2 + 1}{x^2 + 5}$$

$$\frac{x^2 \cdot A}{x^2 \cdot B}$$

quotient rule

$$\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{(v)^2}$$

Example: Find the derivative of $f(x) = \dfrac{\sin(x)}{5x+2}$.

$$f'(x) = \frac{(\cos(x)) \cdot (5x+2) - (\sin(x)) \cdot (5)}{(5x+2)^2}$$

$$f'(x) = \frac{5x \cdot \cos(x) + 2\cos(x) - 5\sin(x)}{(5x+2)^2}$$

a rational function

Example: Find the slope of the tangent line to the curve $f(x) = \dfrac{x^2+x}{x+5}$ at $x=1$.

quotient rule

$$f'(x) = \frac{(2x+1) \cdot (x+5) - (x^2+x) \cdot (1)}{(x+5)^2}$$

$$f'(1) = \frac{3 \cdot 6 - 2 \cdot 1}{(6)^2} = \frac{18-2}{36} = \frac{16}{36} = \frac{4}{9}$$

$$\boxed{m_{tangent} = \frac{4}{9}}$$

Equation of the tangent line: $(1, f(1)) = (1, \frac{2}{6}) = (1, \frac{1}{3})$

$$\boxed{y - \frac{1}{3} = \frac{4}{9}(x-1)}$$

composition of functions

$$(f \circ g)(x) = f(g(x))$$

ex: → $f(x) = \cos(x)$ → $f'(x) = -\sin(x)$

→ $g(x) = x^2$

$$(f \circ g)(x) = f(x^2) = \boxed{\cos(x^2)}$$

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

$$= f'(x^2) \cdot g'(x)$$

$$= \boxed{-\sin(x^2) \cdot 2x}$$

Theorem: The Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composition $f \circ g$ is differentiable at x. Moreover,

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

Case-1: $u^n \xrightarrow{\text{deriv.}} n \cdot u^{n-1} \cdot u'$

This rule is one of the most important rules of differentiation. It helps us with many complicated functions.

Case-1 : $(\text{ugly expression})^n$

Example: Find the derivative of $h(x) = (2x+1)^3$. ← u^3

chain rule

$$h'(x) = 3 \cdot (2x+1)^2 \cdot (2x+1)'$$

$$= 3 \cdot (2x+1)^2 \cdot (2) = \boxed{6(2x+1)^2}$$

Example: Find the derivative of $h(x) = \left(x^3 + 5x + 1\right)^4$. ← u^4

power rule

$$h'(x) = 4 \cdot (x^3 + 5x + 1)^3 \cdot (3x^2 + 5)$$

deriv. of inside

(by chain Rule)

power rule, followed by chain rule

Example: Find the derivative of $h(x)=\left(x^2+\sin(x)\right)^5$.

$$h'(x) = 5\cdot\left(x^2+\sin(x)\right)^4\cdot\left(2x+\cos(x)\right)$$

derivative

Example: Find the derivative of $f(x)=\sin^4(x)$. $= [\sin(x)]^4$

u^4

$$f'(x) = 4\cdot[\sin(x)]^3\cdot(\cos(x))$$

deriv.

$$f'(x) = 4\cdot\sin^3(x)\cdot\cos(x)$$

cos(ugly)

Example: Find the derivative of $f(x)=5\cos\left(x^2+1\right)$.

case-2

$$f'(x) = 5\cdot -\sin(x^2+1)\cdot(2x)$$

deriv. of inside

$$\boxed{f'(x) = -10\cdot x\cdot\sin(x^2+1)}$$

Case-2 Trig functions with "ugly" inside

ex

$f(x) = \cos(x^2)$

chain rule

$f'(x) = -\sin(x^2) \cdot (2x)$

deriv. of inside

ex: $g(x) = \sin(\underbrace{x^3}_{\text{ugly}})$

$g'(x) = \cos(x^3) \cdot (3x^2)$

deriv. of inside

Case-2

$$\sin(5x) \xrightarrow{\text{deriv.}} 5 \cdot \cos(5x)$$

$$\cos(4x) \xrightarrow{\text{deriv.}} -4 \cdot \sin(4x)$$

$$\tan(3x) \xrightarrow{\text{deriv.}} 3 \cdot \sec^2(3x)$$

$$\text{trig}(m \cdot x)$$

$\cos(\underbrace{5x}) \xrightarrow{\text{der}} -5\sin(\underbrace{5x})$

Remark: Note that, using chain rule, we have the following results. You will see such questions very often; familiarize yourself with these types of questions.

$$\left[\sin(mx)\right]' = m\cos(mx)$$

$$\left[\cos(mx)\right]' = -m\sin(mx)$$

$$\left[\tan(mx)\right]' = m\sec^2(mx)$$

$[\sec(x)]' = \sec(x)\cdot\tan(x)$

$[\sec(mx)]' = m\cdot\sec(mx)\cdot\tan(mx)$

Example: Find the derivative of $h(x) = \underbrace{\cos(4x)} + 2\underbrace{\sin(7x)}$.

$h'(x) = -4\sin(4x) + 2\cdot 7\cdot\cos(7x)$

$h'(x) = -4\sin(4x) + 14\cos(7x)$

what if $h'(0) = -4\sin(0) + 14\cos(0) = 0+14$

$= \boxed{14}$

Example: Find the derivative of $g(x) = x^2\tan(4x)$.

$\boxed{x^2}\cdot\boxed{\tan(4x)}$

$\uparrow u$ $\quad$ $\uparrow v$

product rule

$g'(x) = (\boxed{2x})\cdot\tan(4x) + x^2\cdot\boxed{(4\cdot\sec^2(4x))}$

$g'(x) = 2x\tan(4x) + 4x^2\cdot\sec^2(4x)$

a combination of case-1 & case-2

In general, $f(x)=\underbrace{\sin}_{=}(\underbrace{g(x)}) \to f'(x)=\underbrace{g'(x)}_{=}\underbrace{\cos(g(x))}_{=}$.

Example: Find the derivative of $f(x)=\sin^3(5x^2)$. $= [\sin(5x^2)]^3$

$$f'(x) = 3\cdot[\underbrace{\sin(5x^2)}]^2\cdot(\cos(5x^2)\cdot 10.x)$$

deriv. of ..

$$f'(x) = 30x\cdot\sin^2(5x^2)\cdot\cos(5x^2)$$

Example: Find the derivative of $f(x)=5\cos^2(x^2+4)$.

$$f(x) = 5\cdot[\cos(x^2+4)]^2$$

$$f'(x) = 5\cdot 2[\underbrace{\cos(x^2+4)}]\cdot(-\sin(x^2+4)\cdot 2x)$$

deriv.

$$f'(x) = -20x\cdot\cos(x^2+4)\cdot\sin(x^2+4)$$

$u^n \longrightarrow n \cdot u^{n-1} \cdot u'$

Rational Powers and Chain rule: $\sqrt{ugly} = u^{1/2}$

Example: Find the derivative of $f(x) = \sqrt[3]{4x+7}$. $= (4x+7)^{1/3}$

$\frac{1}{3} - 1 = -\frac{2}{3}$

$f'(x) = \frac{1}{3} \cdot (4x+7)^{-2/3} \cdot (4) = \boxed{\frac{4}{3}(4x+7)^{-2/3}}$

deriv. of inside

Example: Find the derivative of $f(x) = \sqrt{2+\cos(5x)}$.

$\sqrt{ugly} = u^{1/2} \xrightarrow{\text{deriv.}} \frac{1}{2} \cdot u^{-1/2} \cdot u'$

$*$ $\boxed{(\sqrt{u})' = \frac{u'}{2\sqrt{u}}}$

$f(x) = (2+\cos(5x))^{1/2} = \sqrt{2+\cos(5x)}$

$\boxed{f'(x) = \frac{-5\sin(5x)}{2\sqrt{2+\cos(5x)}}}$

$$\cos(x) \xrightarrow{\text{deriv}} -\sin(x)$$

$$\cos(\underline{5}x) \xrightarrow{\text{deriv}} -\underline{5}\sin(5x)$$

$$\begin{array}{ccc} 2 & + & \cos(5x) \\ \downarrow & & \downarrow \\ 0 & + & \left(-5\sin(5x)\right) \end{array}$$

$$\left(2+\cos(5x)\right)^{1/2}$$

$$\frac{1}{2}\left(2+\cos(5x)\right)^{-1/2} \cdot (-5\sin(5x))$$

Attendance

Day 7

4 — A

5 — B

6 — C

7 — D

8 — E

Labquiz

2.1 & 2.2

Exercise: Find the derivative of $f(x) = \sin(x)\sqrt{1+\cos(2x)}$. Also, compute $f'\left(\frac{\pi}{4}\right) = ?$

IMPORTANT Example: The following information is given about two functions f and g.

$$f(1)=6,\ f'(1)=4,\ f(7)=2,\ f'(7)=1,$$

$$g(1)=7,\ g'(1)=8,\ g(6)=10,\ g'(6)=2.$$

a) If $h(x)=(f\circ g)(x)$, find $h'(1)$.

b) If $h(x)=(fg)(x)$, find $h'(1)$.

c) If $h(x)=\left(\dfrac{f}{g}\right)(x)$, find $h'(1)$.

Exercises: Previous problem, continued:

d) If $h(x)=\left[f(x)\right]^3$, find $h'(1)$.

e) If $h(x)=(g\circ f)(x)$, find $h'(1)$.

The Chain Rule in Leibniz Notation

This is what the chain rule says with Leibniz's double-d notation:

$$\frac{d}{dx}\left[f\left(u(x)\right)\right] = f'\left(u(x)\right)\cdot u'(x) \quad \text{or} \quad \frac{d}{dx}\left[f(u)\right] = f'(u)\cdot\frac{du}{dx}.$$

If $y = f(u)$, then

$$\frac{dy}{dx} = f'(u)\cdot\frac{du}{dx}$$

and since $f'(u) = \frac{dy}{du}$, the chain rule can be written as:

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}.$$

That is, the derivative of y with respect to x is the product of the derivative of y with respect to u and the derivative of u with respect to x.

This formula can be extended to more variables; each new variable adds a new link to the chain.

For the composition of 3 functions,

$$\frac{d}{dx}\left[f\left(u\left(v(x)\right)\right)\right] = f'\left(u\left(v(x)\right)\right)\cdot u'\left(v(x)\right)\cdot v'(x)$$

can be written as:

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}.$$

Example: If $y = 5u + 1$ and $u = 6x^2$, evaluate $\left.\frac{dy}{dx}\right|_{x=1}$.

Option 1: Direct substitution

$$y = 5u + 1 \longrightarrow y = 5\left(6x^2\right) + 1 = 30x^2 + 1$$

$\frac{dy}{dx} = 60x$; that is $\left.\frac{dy}{dx}\right|_{x=1} = 60 \cdot 1 = 60$

Option 2: Use Chain rule with Leibniz Notation

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (5)(12x) = 60x$$

$$\left.\frac{dy}{dx}\right|_{x=1} = 60$$

In cases where substitution might give complicated expressions, the second option might come handy.

Exercise: If $y = u^2$ and $u = 2x - 6$, $x = t^2 + 1$, evaluate $\left.\frac{dy}{dt}\right|_{t=2}$.

Hint: when $t = 2$, we have: $x = 2^2 + 1 = 5$, and $u = 2(5) - 6 = 4$.

Chain rule: $\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dx} \cdot \frac{dx}{dt}$