Completed notes

Exercises solved!

Check extra examples included!

Math 2413- Calculus I

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- Check CASA calendar for due dates.
- Bring "blank notes" to class. Completed notes will be posted after class.
- Do your best to attend every lecture and lab.
- Study after every lecture; work on the quiz covering the topic we cover on the lecture immediately afterwards. Retake your quizzes for more practice.
- Get help when you need help; bring your questions to the labs, or my office hours. We also have tutoring options on campus.
- Respect your friends in class; stay away from distractive behavior. Do your best to concentrate on the lecture.
- If you email me, mention the course code in the subject line. Email is the best way to communicate with me outside of class. Teams chat messages are not monitored or replied to.

Chapter 3

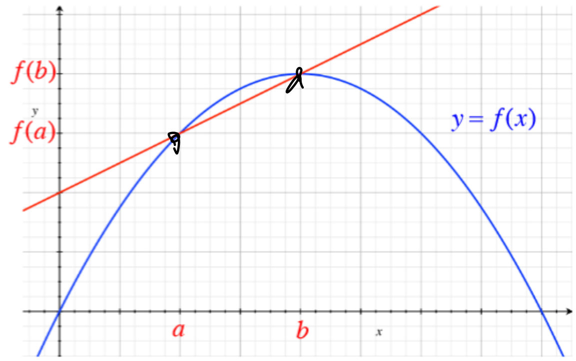
Section 3.1 – Related Rates

The Derivative as a Rate of Change

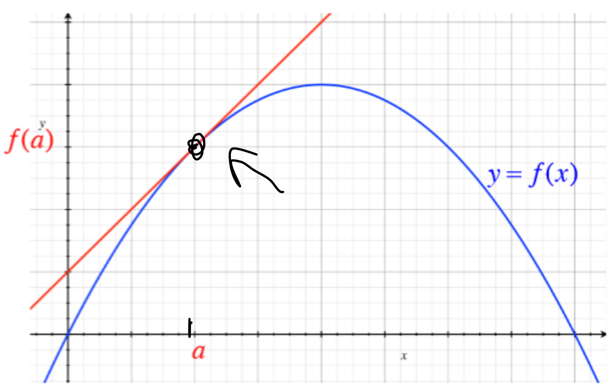
Given $y = f(x)$, the slope of the secant line

$$\frac{f(b)-f(a)}{b-a}$$

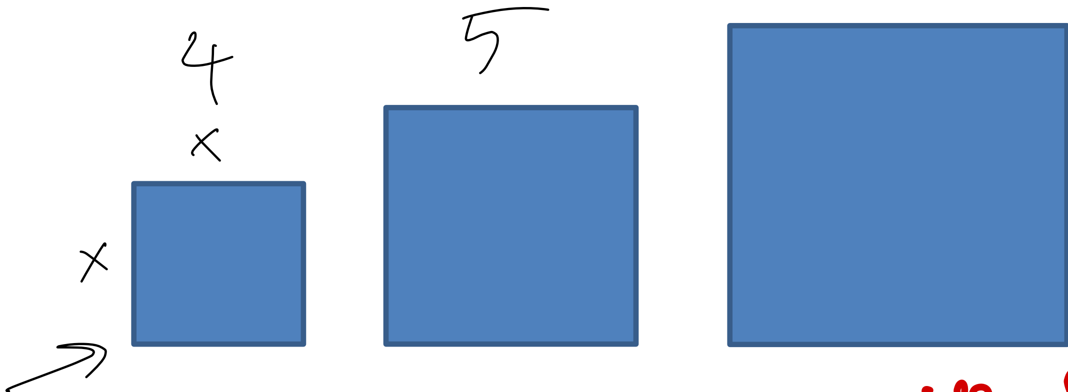
gives the **average rate of change** in y with respect to x over the interval $[a,b]$.



On the other hand, the slope of the tangent line, $f'(a)$, gives the **instantaneous rate of change** in y with respect to x at a. The instantaneous rate of change is the limit of the average rates of change as the interval width approaches 0.



Example: The side length of a square is changing.



Let's say $A = area$ and $x = side\ length$.

s: side length

We know the area formula: $A = x^2$

$A = s^2$

$\frac{dA}{ds} = 2 \cdot s$

The rate of change of the area with respect to side length is:

$$\frac{dA}{dx} = 2 \cdot x$$

Leibniz notation

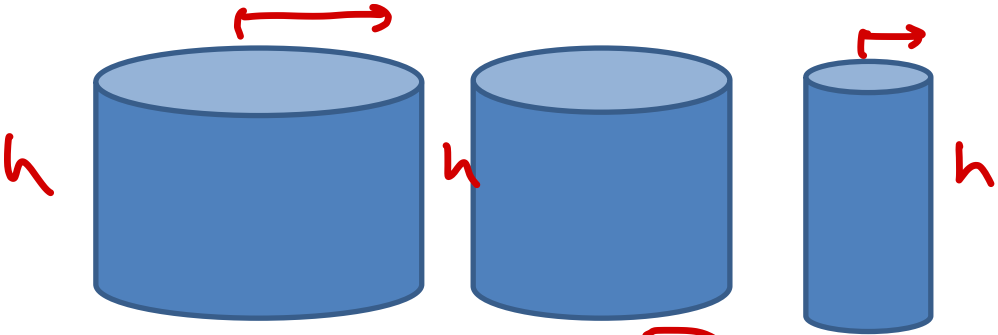
How fast is the area changing when $x = 5$?

$$\left.\frac{dA}{dx}\right|_{x=5} = 2 \cdot 5 = 10$$

$A = x^2$

$\frac{dA}{dx} = 2x$

Example: The radius of a right cylinder is decreasing while the height remains constant.



$V = \pi r^2 h$ $\qquad \pi \cdot r^2 \cdot h$

What is the rate of change of volume **with respect to radius?**

$$\frac{dV}{dr} = \pi \cdot 2 \cdot r \cdot h$$

$V = \pi \cdot r^2 \cdot h$

What is the rate of change of volume **with respect to time? (Radius is changing with respect to time, height is constant)**

$$\frac{dV}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt} \cdot h$$

$r \rightarrow t$

$\pi \cdot r^2 \cdot h$

Extra: What if both h and r were changing with respect to time? product rule

$V = \pi r^2 h$

$$\frac{dV}{dt} = \underbrace{\pi \cdot 2r \cdot \frac{dr}{dt}}_{u'} \cdot h + \pi \cdot r^2 \cdot \underbrace{\frac{dh}{dt}}_{v'}$$

Section 3.1 – Related Rates

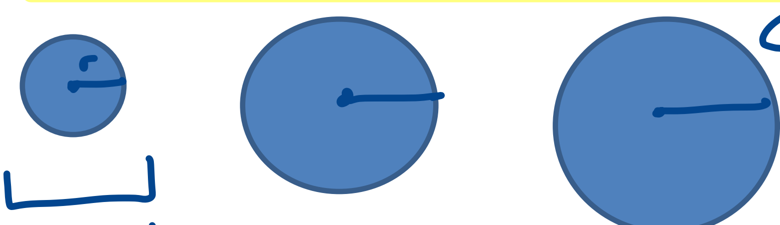
You should know these formulas from Geometry:

> Volume of a sphere, cone, cylinder, prism, cube.
> Area of a circle, rectangle, triangle
> Surface area of a cube, right cylinder, prism.
> Pythagorean Theorem.

Related Rates:

- Draw a "picture".
- What do you know?
- What do you need to find?
- Write an equation involving the variables whose rates of change either are given or are to be determined. (This is an equation that relates the parts of the problem.)
- Implicitly differentiate both sides of the equation with respect to time. This FREEZES the problem.
- Solve for what you need.

Example 1: Assume that oil spilled from a ruptured tanker spreads in a circular pattern whose radius increases at a constant rate of 2 ft/sec. How fast is the area of the spill increasing when the radius of the spill is 60 feet?



Given: $\frac{dr}{dt} = 2$ ft/sec ↑

r: radius

freeze $r = 60$

$$\boxed{A = \pi \cdot r^2}$$

implicit diff.

$$\frac{dA}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt}$$

related rates

Question : $\left.\dfrac{dA}{dt}\right|_{r=60\,ft} = ?$

$$\left.\frac{dA}{dt}\right|_{r=60} = \pi \cdot 2 \cdot \underset{ft}{(60)} \cdot \underset{ft/sec}{2}$$

$$\left.\frac{dA}{dt}\right|_{r=60} = 240\pi \quad ft^2/sec$$

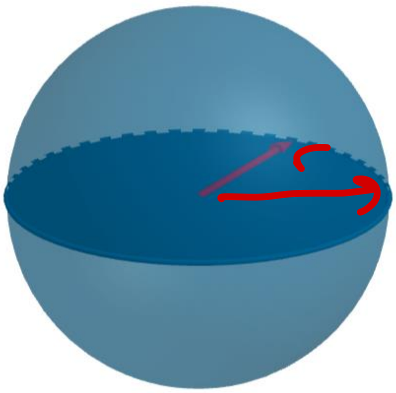
$$f(x) = \pi \cdot x^2 + \pi$$

$$f'(x) = \pi \cdot 2x + 0$$

keep

Example 2: Suppose a spherical balloon is inflated at the rate of 10 cubic centimeters per minute. How fast is the radius increasing when the radius is 5 centimeters?
https://www.geogebra.org/m/PgtxDUCQ

Given: $\frac{dV}{dt} = 10\ cm^3/min$



Question: $\left.\frac{dr}{dt}\right|_{r=5} = ?$

Relationship: $V = \frac{4}{3}\pi \cdot r^3$

implicit diff (wrt time)

$$\frac{dV}{dt} = \frac{4}{\cancel{3}} \cdot \pi \cdot \cancel{3} \cdot r^2 \cdot \frac{dr}{dt}$$

$$10 = 4 \cdot \pi \cdot r^2 \cdot \frac{dr}{dt}$$

r ?

$$10 = 4 \cdot \pi \cdot (5)^2 \cdot \left.\frac{dr}{dt}\right|_{r=5}$$

?

$$10 = 7 \cdot A \qquad A = \frac{10}{7}$$

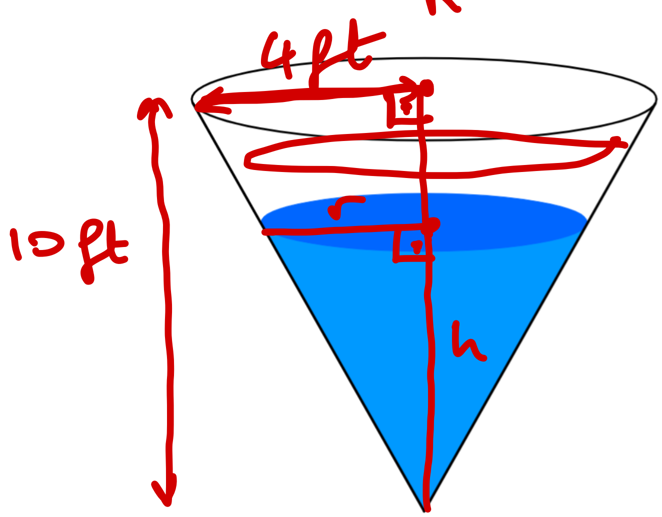
$$10 = 100\pi \cdot A \Rightarrow A = \frac{1\cancel{0}}{10\cancel{0}\pi}$$

$$= \frac{1}{10\pi}$$

$$\Rightarrow \frac{dr}{dt}\Big|_{r=5} = \frac{10}{4\cdot\pi\cdot 25} = \boxed{\frac{1}{10\pi}}$$

cm/min

Example 3: Water is pouring into an inverted cone shaped tank at the rate of 20 ft^3/min. The tank is 10 ft. tall and has a radius of 4 ft. How fast is the height of the water rising when it is 5 ft deep?



Given: $\frac{dV}{dt} = 20 \text{ ft}^3/\text{min}$

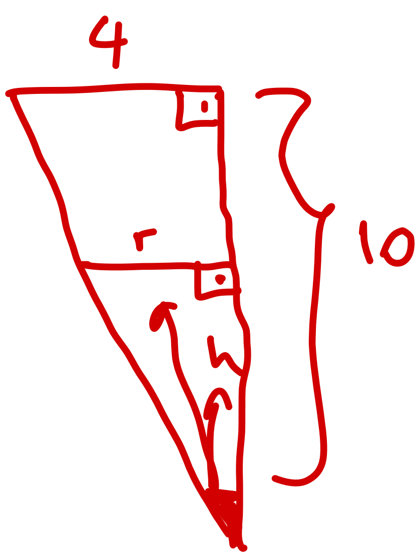
volume/time

Question: $\frac{dh}{dt}\Big|_{h=5 \text{ ft}} = ?$

$$V = \frac{1}{3}\cdot\pi\cdot r^2\cdot h$$

substitute

implicit differentiation (wrt time)



$\frac{r}{h} = \frac{\cancel{4}}{\cancel{10}} = \frac{2}{5}$ cross product

$10r = 4h$

$5r = 2h \Rightarrow r = \frac{2}{5}h$

$$V = \frac{1}{3}\cdot\pi\cdot\left(\frac{2h}{5}\right)^2\cdot h = \frac{1}{3}\cdot\frac{4}{25}\cdot\pi\cdot h^3$$

$$V = \frac{4}{75}\pi\cdot h^3$$

Implicit diff. (dt)

$$\boxed{\frac{dV}{dt}} = \frac{4}{\cancel{75}\ 25}\pi \cdot \cancel{3}h^2 \cdot \boxed{\frac{dh}{dt}}$$

20

$$20 = \frac{4}{25}\pi \cdot h^2 \cdot \frac{dh}{dt} \ ?$$

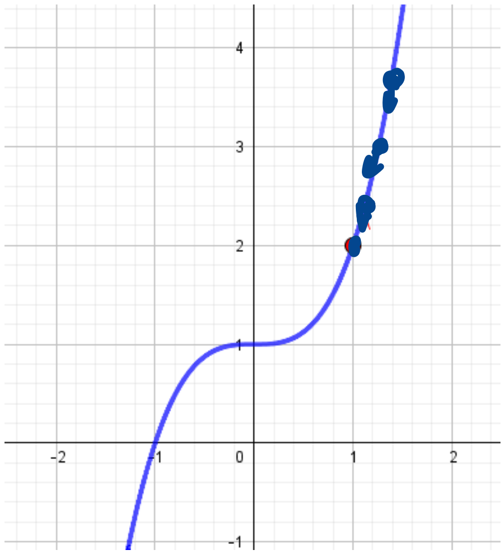
$$\frac{dh}{dt} = \frac{20 \cdot 25}{4 \cdot \pi \cdot \underline{\underline{h}}^2}$$

plug in $h=5$

$$\left.\frac{dh}{dt}\right|_{h=5} = \frac{20 \cdot 25}{4 \cdot \pi \cdot (5)^2}$$

$$= \boxed{\frac{5}{\pi}} \text{ ft/min}$$

Example 4: A point moves along the curve $y = x^3 + 1$ in such a way that the y value is decreasing at the rate of 2 units per second. At what rate is x changing when $x = 1$?



Given: $\frac{dy}{dt} = -2$ units/sec.

reason: decreasing

Question: $\left.\frac{dx}{dt}\right|_{x=1} = ?$

Relationship: $y = x^3 + 1$

Implicit diff. wrt. the:

$$\underbrace{\frac{dy}{dt}}_{\text{given}} = 3 \cdot x^2 \cdot \frac{dx}{dt}$$

$$-2 = 3 \cdot x^2 \cdot \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{-2}{3x^2}$$

$$\left.\frac{dx}{dt}\right|_{x=1} = \frac{-2}{3 \cdot 1} = \boxed{\frac{-2}{3}}$$ units/sec

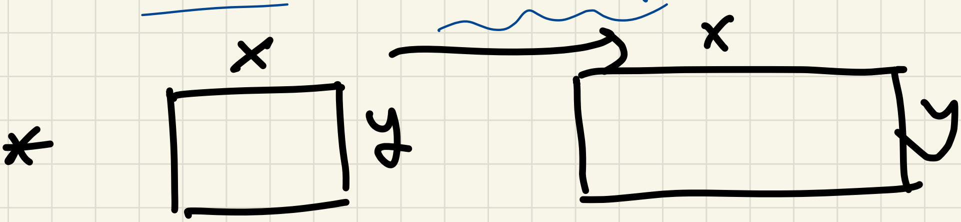
WHW5 : Review sheet
due on Tuesday

Practice Test 1 : extra credit
& a good way to
prepare for your test

In-class review: Next Tuesday.

* Lectures & labs continue
during the testing window.

§3.1 Recap



$$A = x \cdot y$$

x: changes with time

y: stays the same

$$\frac{dA}{dt} = ?$$

$$\frac{dA}{dt} = \frac{dx}{dt} \cdot y$$

---

y: changing   x: const

or

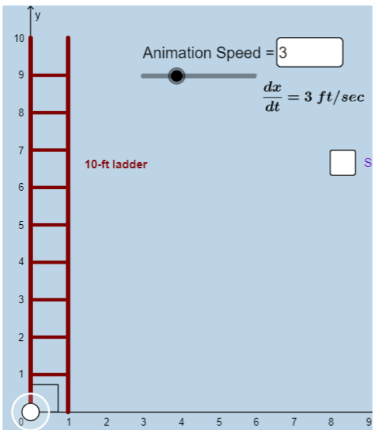
$$\frac{dA}{dt} = x \cdot \frac{dy}{dt}$$

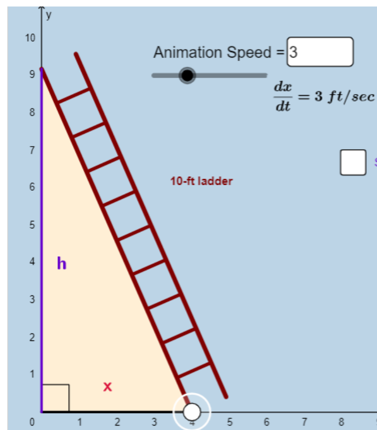
$A = x \cdot y$ both x & y changing

$$\frac{dA}{dt} = \frac{dx}{dt} \cdot y + x \cdot \frac{dy}{dt}$$

Example 5: A 10-foot ladder, leaning against a wall, slips so that its base moves away from the wall at a rate of 3 ft/sec. How fast will the top of the ladder be moving down the wall when the base is 4 feet from the wall?

https://www.geogebra.org/m/ffAEhyNk

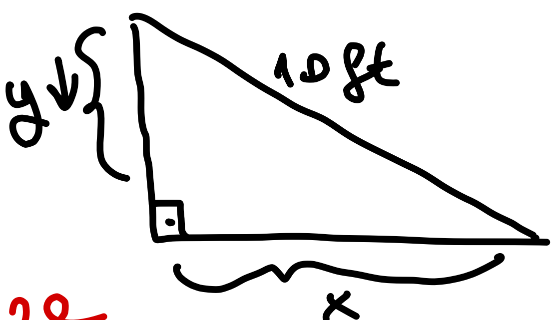




Given:

$$\frac{dx}{dt} = 3\ ft/sec$$

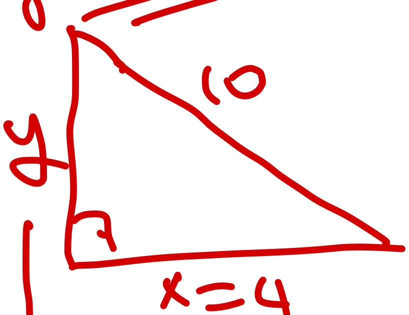
Q: $\left.\frac{dy}{dt}\right|_{x=4} = ?$



$$x^2 + y^2 = 100$$

implicit diff (t)

freeze



$$y = \sqrt{10^2 - 4^2}$$

$$y = \sqrt{84}$$

$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$$

$$2 \cdot 4 \cdot 3 + 2 \cdot \sqrt{84} \cdot \boxed{\frac{dy}{dt}} = 0$$

$$\frac{dy}{dt} = \frac{-24}{2\sqrt{84}} = \frac{-12}{\sqrt{84}}$$

$$\frac{dy}{dt} = \frac{-12}{2\sqrt{21}} = \frac{-6}{\sqrt{21}}$$

$$\frac{dy}{dt} = \frac{-6\sqrt{21}}{21} = \boxed{-\frac{2\sqrt{21}}{7}}$$

Homer's Shadow:

https://www.geogebra.org/m/UVJAPXVK

Shadow problem (walking away from the lamp):

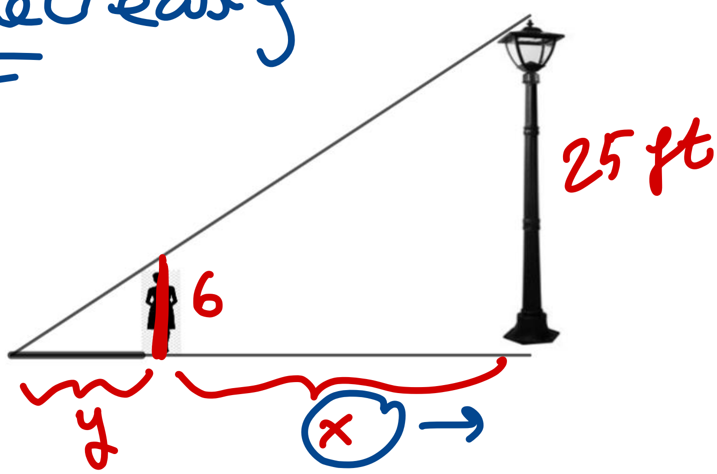
https://www.geogebra.org/m/djVFxBbX

Example 6: A 6-foot person is walking towards a 25 foot lamp post at the rate of 10 feet per second. How fast is the length of their shadow changing when the person is 20 feet from the lamp post?

x is decreasing

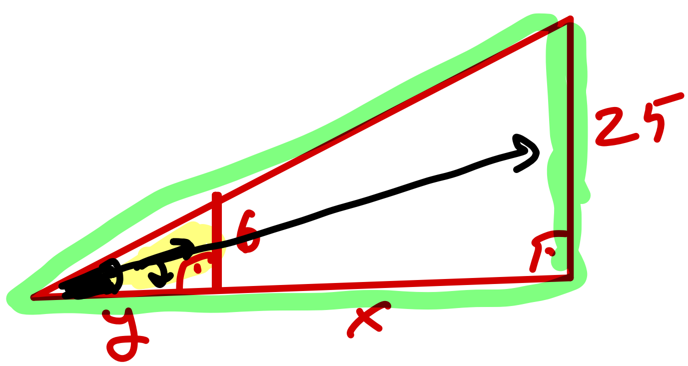
x: decreasing

Given: $\frac{dx}{dt} = -10$ ft/sec



Question: $\left.\frac{dy}{dt}\right|_{x=20} = ?$

$$\frac{6}{y} = \frac{25}{x+y}$$

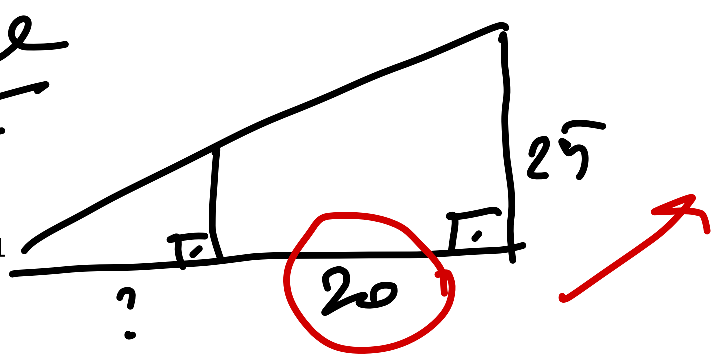


$$6x + 6y = 25y$$

$$6x = 19y$$

↓ deriv. (t)

freeze



$$6 \cdot \frac{dx}{dt} = 19 \cdot \frac{dy}{dt}$$

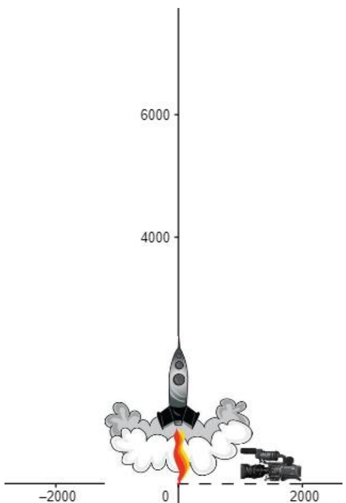
$$6 \cdot -10 = 19 \cdot \frac{dy}{dt}$$

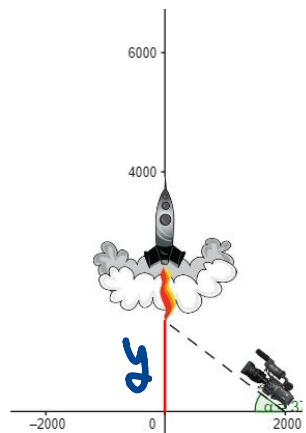
$$\Rightarrow \quad \frac{dy}{dt} = \frac{-60}{19} \text{ ft/sec}$$

(decreasing)

Rocket Rising:
https://www.geogebra.org/m/baQCm7UH

Example 7: If a rocket is rising vertically at the rate of 1200 ft/sec when it is 4000 feet up, how fast is the camera-to-rocket distance changing at the instant?





$$\frac{dy}{dt} = 1200\,ft/sec$$

$$\left.\frac{dz}{dt}\right|_{y=4{,}000} = ?$$

freeze rocket

4000 ft

y

z

$\boxed{2{,}000}$

camera

$$\underbrace{(2{,}000)^2}_{} + \underbrace{y^2}_{} = z^2$$

$$0 + 2y \cdot \frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$\underbrace{y \cdot \frac{dy}{dt}}_{} = \underbrace{z \cdot \frac{dz}{dt}}_{?}$$

4000

2000

$$\sqrt{16 \cdot 10^6 + 4 \cdot 10^6}$$

$$\sqrt{20 \cdot 10^6}$$

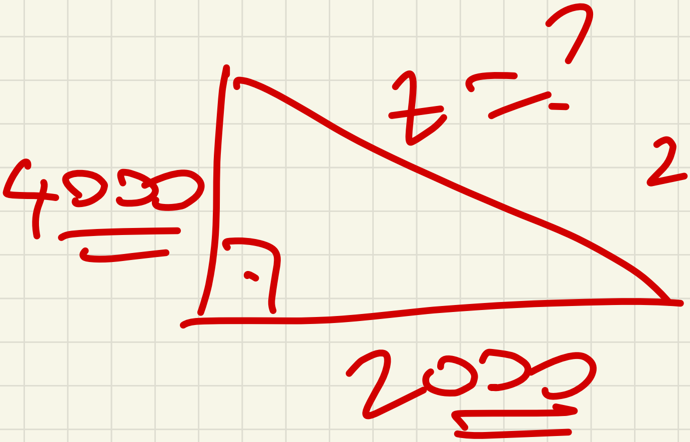
$$z = 2\sqrt{5} \cdot 10^3$$

$$\Rightarrow \quad 4{,}000 \cdot 1200 = 2\sqrt{5} \cdot \cancel{10^3} \cdot \frac{dz}{dt}$$

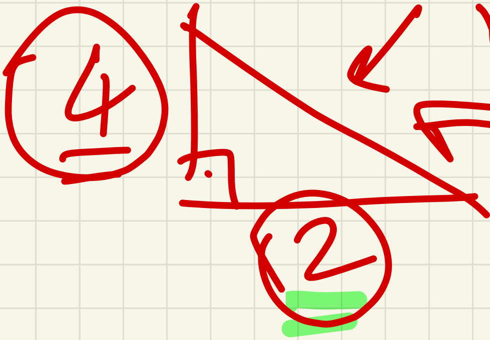
$$\Rightarrow \quad \frac{dz}{dt} = \frac{4 \cdot 1200}{2\sqrt{5}}$$

$$\frac{dz}{dt} = \frac{2400}{\sqrt{5}} = \frac{2400\sqrt{5}}{5}$$

$$\boxed{\frac{dz}{dt} = 480\sqrt{5}}$$



$2\sqrt{5} \cdot 1000$



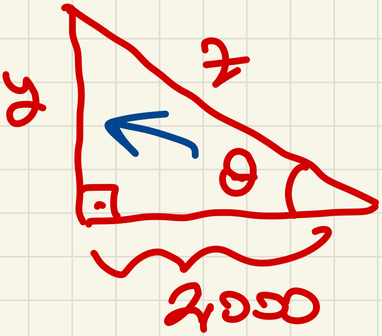
$\sqrt{16+4} = \sqrt{20}$

$2\sqrt{5}$

what if

Q: how fast is the angle changing?

$$\frac{d\theta}{dt} = ?$$



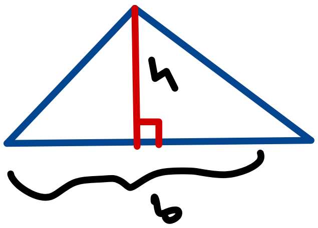
$$\tan(\theta) = \frac{y}{2000}$$

deriv. $\Rightarrow$ $$\sec^2(\theta) \cdot \frac{d\theta}{dt} = \frac{1}{2000} \cdot \frac{dy}{dt}$$

??

given: $\frac{dh}{dt} = 2, \frac{db}{dt} = 3$

Exercise: The altitude and base of a triangle is changing at a constant rate of 2 in/sec and 3 in/sec respectively. How fast is the area changing when base is 10 inches and altitude is twice the base?



$A = \frac{1}{2} \cdot b \cdot h$

Both b&h are changing

$$\frac{dA}{dt} = \frac{1}{2} \cdot \frac{db}{dt} \cdot h + \frac{1}{2} \cdot b \cdot \frac{dh}{dt}$$

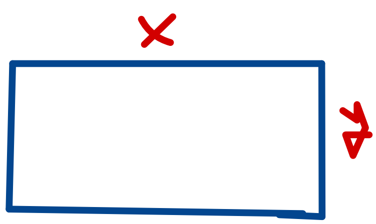
given given given

freeze when $b=10$ & $h=20$

$$\frac{dA}{dt} = \frac{1}{2} \cdot 3 \cdot 20 + \frac{1}{2} \cdot 10 \cdot 2$$

$$\frac{dA}{dt} = 30 + 10 = \underline{\underline{40}} \text{ in}^2/\text{sec}$$

Exercise: The length of a rectangle is increasing at a rate of 2 in/min while the width is decreasing at a rate of 3 in/min. How fast is the area changing when length is 20 and width is 5 inches?



$\frac{dx}{dt} = 2$ in/min (increasing: +)

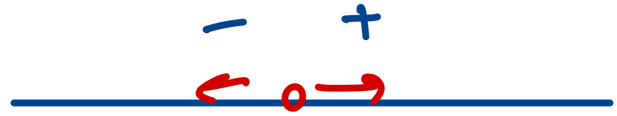
$\frac{dy}{dt} = -3$ in/min (decreasing: −)

$$A = x \cdot y \xRightarrow[\text{both changing}]{\text{deriv.}} \frac{dA}{dt} = \frac{dx}{dt} \cdot y + x \cdot \frac{dy}{dt}$$

given given

$$\left.\frac{dA}{dt}\right|_{\substack{x=20 \\ y=5}} = (2) \cdot 5 + 20 \cdot (-3)$$

$$= 10 - 60 = \underline{\underline{-50}} \text{ in}^2/\text{min}$$



$s(t) = t^2 + 1$

Position, Velocity, Acceleration

Velocity

Imagine a particle moving along a straight-line path in some way. On this line, choose a point of reference, a positive direction and a negative direction. For example, we can choose the line to be the $x-$axis, reference point can be the origin, moving to the right is the positive direction, and left is the negative direction.

Let t be a variable representing the time elapsed since some reference time (when $t=0$). Let $s(t)$ be the position of the particle at time t measured relative to some reference point (where $s=0$). If the position function $s(t)$ is differentiable, then the derivative $s'(t)$ gives the **rate of change** of the position function at time t. This rate is called the **velocity at time** t and denoted as $v(t)$. In symbols,

$$v(t) = s'(t).$$

Average velocity over a time interval $a \le t \le b$ is the change in position divided by the change in time:

$$v_{avg} = \frac{\Delta s}{\Delta t} = \frac{s(b)-s(a)}{b-a}.$$

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Acceleration

Acceleration is defined to be the rate of change of velocity per unit time. If the velocity function $v(t)$ is differentiable, then its derivative gives the acceleration function;

$$a(t) = v'(t).$$

Since, $v(t)$ is the derivative of $s(t)$, acceleration is the second derivative of position:

$$a(t) = v'(t) = s''(t) \qquad \text{or} \qquad a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}.$$

Here are the connections between position, velocity, acceleration and speed:

- Positive velocity corresponds to motion in positive direction (position is increasing). Negative velocity corresponds to motion in negative direction (position is decreasing).
- Positive acceleration corresponds to increasing velocity. Negative acceleration corresponds to decreasing velocity.

speed = |velocity|

- The object is **speeding up** if the velocity and acceleration have the same sign. The object is **slowing down** if the velocity and acceleration have opposite signs.

$v(t) = 2t - 6$

$a(t) = v'(t) = 2$

Example: An object moves along the x – axis and its position is given by the function

$$s(t)=t^3-9t^2+15t+8.$$

$3(t^2-6t+5)$

a) Find the velocity of this object at time t = 2.

$v(t) = s'(t) = 3t^2 - 18t + 15$

$v(2) = 3\cdot 4 - 18\cdot 2 + 15 = -9$ speed: $\underline{\underline{9}}$

b) Find the acceleration at time t = 2.

$a(t) = v'(t) = 6t - 18$

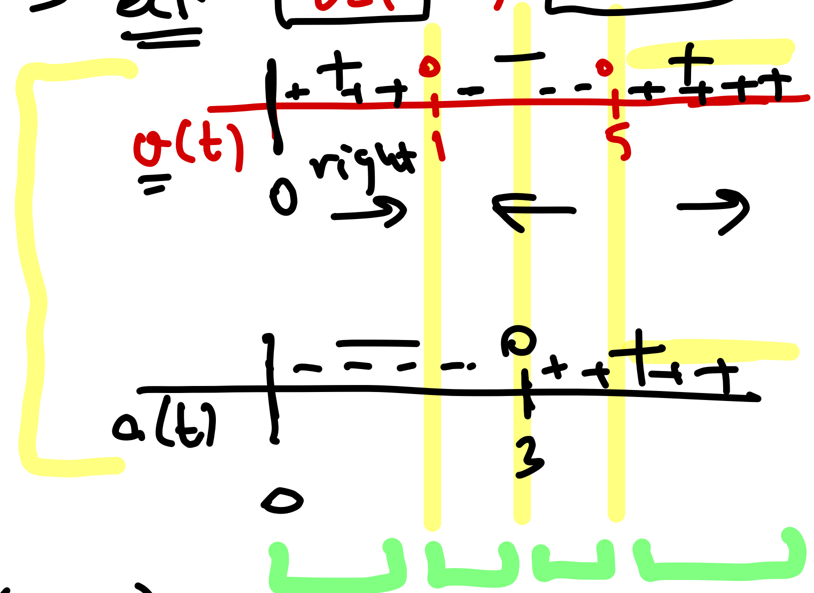
$a(2) = 6\cdot 2 - 18 = -6$

c) When does this object change direction?

sign chart for $v(t) = \boxed{3(t-1)(t-5)}$

d) When is the object speeding up? at $\boxed{t=1}$, $\boxed{t=5}$

e) When is it slowing down?



Sign of $a(t)$ & $v(t)$

$a(t) = 6(t-3)$

(0,1) slowing down (+, −)

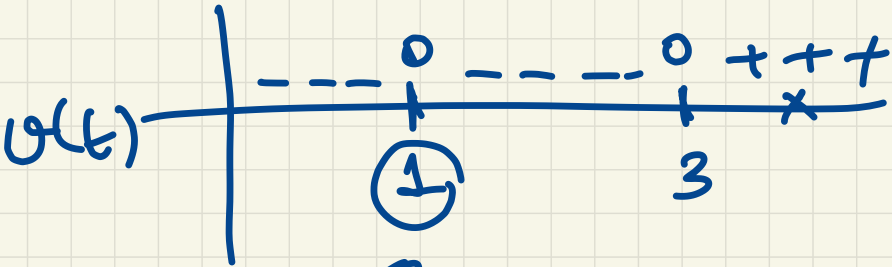
(1,3) speeding up (−, −)

(3,5) slowing down (−, +)

(5, ∞) speeding up (+, +)

ex

$$v(t) = 5(t-1)^{2}(t-3)$$



double root

sign chart

solving inequalities using sign charts

Exercise: If x(t) = ½ t^4 – 5t^3 + 12t^2, find the velocity of the moving object when its acceleration is zero.

Free Fall Formulas

The height of an object in free fall is given by

$$h(t) = -16t^2 + v_0 t + h_0 \quad \text{(distance in feet)}$$

or

$$h(t) = -4.9t^2 + v_0 t + h_0 \quad \text{(distance in meters),}$$

extra

where v_0 is the initial velocity and is the initial height.

$h(t) = -16t^2 + \cancel{v_0} \cdot t + h_0$

Example: An object is dropped from a height of 320 feet.

position $\rightarrow h(t) = -16t^2 + 320$

a) What is its height after 1 second?

b) What is its velocity at time t=1?

$v(t) = h'(t) = -32t$

$v(1) = -32$

c) How long does it take for the object to hit the ground?

$h(t) = 0 \Rightarrow -16t^2 + 320 = 0 \Rightarrow t^2 = \frac{320}{16} = 20$

$\Rightarrow t = \sqrt{20} = 2\sqrt{5}$
$(t > 0)$

d) What is the speed on impact?

speed when $t = 2\sqrt{5}$

$v(t) = -32t$ $\quad v(2\sqrt{5}) = -32 \cdot 2\sqrt{5}$
$= -64\sqrt{5}$

speed on impact: $|-64\sqrt{5}| = \boxed{64\sqrt{5}}$

means when object hits the ground.

Example: A stone, projected upward with an initial velocity of 112 ft/sec, moves according to $x(t) = -16t^2+112t$.

a) Compute the velocity and acceleration when $t = 3$ and when $t = 4$.

$x(t) = -16t^2 + 112t$

$v(t) = -32t + 112$

$v(3) = -32 \cdot 3 + 112 = 16$ ft/sec ($\uparrow$ since >0)

$v(4) = -32 \cdot 4 + 112 = -16$ ft/sec ($\downarrow$ since <0)

b) Determine the greatest height the stone will reach.

greatest height: max value for $x(t) = -16t^2+112t$

$v(t) = -32t + 112 = 0$

$\Rightarrow t = \frac{112}{32} = \frac{7}{2}$ sec.

max height: $x\left(\frac{7}{2}\right) = -16\left(\frac{7}{2}\right)^2 + 112 \cdot \frac{7}{2}$

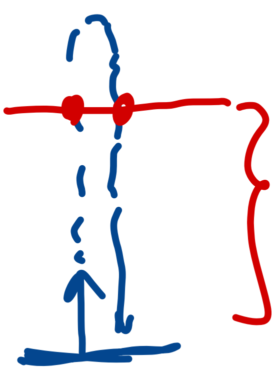
c) Determine when the stone will have a height of 96 ft.

$x(t) = 96 \Rightarrow -16t^2 + 112t = 96$

$-16t^2 + 112t - 96 = 0$

$-16(t^2 - 7t + 6) = 0$

$-16(t-6)(t-1) = 0$



answer $\rightarrow$ $\boxed{t=1, \ t=6}$

3.1 is over

<u>Exercise:</u> Supplies are dropped from a stationary helicopter and seconds later hit the ground at 98 meters per second. How high was the helicopter?

EXTRA Examples (Exercises for you!)

Ex: $y(t) = -16\,t^2 + y_0$. The object hits the ground in 4 seconds. What is the initial height?

Ex: $y(t) = -16\,t^2 + 80\,t$.

a) How long does it take to reach the maximum height? What is the maximum height?

b) When does the object reach 64 feet?

c) What is the speed of this object when the height is 64 feet?

Ex: $y(t) = -16\,t^2 + 160t$.

a) What is the height of this object after 2 seconds?

b) What is the velocity after 2 seconds?

c) When does the object reach velocity 80 ft/ sec?

d) When does the object reach height 80 ft?

e) What is the speed of this object when the height is 144 ft.?

f) When does the object hit the ground?

g) What is the speed on impact when it hits the ground?