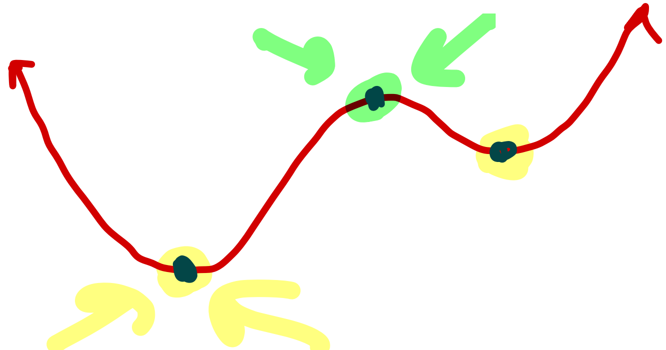
Math 2413- Calculus I

Dr. Melahat Almus

Email: malmus@uh.edu

- Check CASA calendar for due dates.
- Bring "blank notes" to class. Completed notes will be posted after class.
- Do your best to attend every lecture and lab.
- Study after every lecture; work on the quiz covering the topic we cover on the lecture immediately afterwards. Retake your quizzes for more practice.
- Get help when you need help; bring your questions to the labs, or my office hours. We also have tutoring options on campus.
- Respect your friends in class; stay away from distractive behavior. Do your best to concentrate on the lecture.
- If you email me, mention the course code in the subject line. Email is the best way to communicate with me outside of class. Teams chat messages are not monitored or replied to.



Section 3.4 – Extreme Values

Local Extreme Values

Definition: Suppose that f is a function defined on open interval I and c is an interior point of I.

The function f has a local **minimum** at $x = c$ if

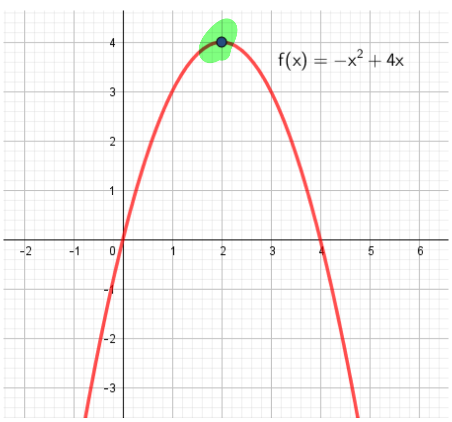
$$f(c) \le f(x) \text{ for all } x \text{ in } I \text{ (that is, for all } x \text{ sufficiently close to } c\text{).}$$

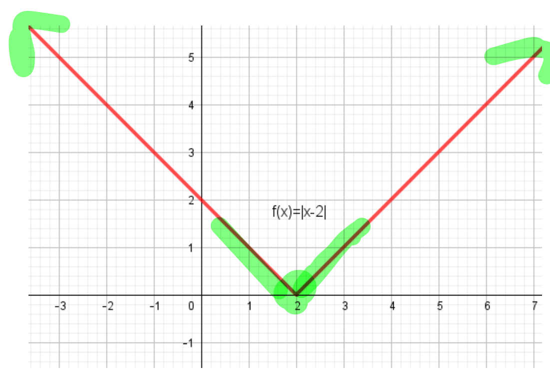
The function f has a local **maximum** at $x = c$ if

$$f(c) \ge f(x) \text{ for all } x \text{ in } I \text{ (that is, for all } x \text{ sufficiently close to } c\text{).}$$

In general, if f has a local minimum or maximum at $x = c$, we say that $f(c)$ is a **local extreme value** of f.

Examples:





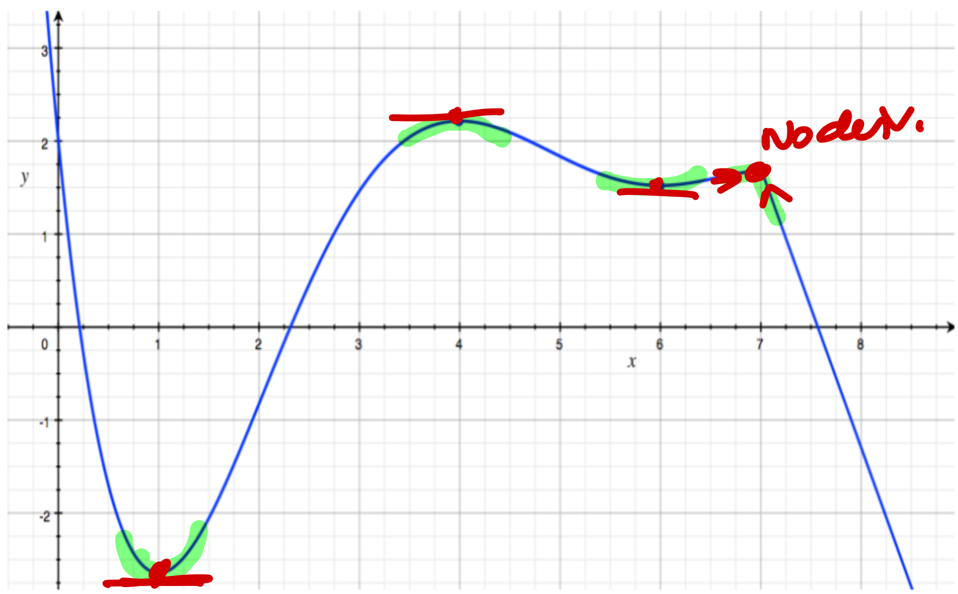
Max value at $x = 2$; max value is 4.

No min value

No max value

Min value at $x = 2$; min value is 0.

When the graph of a function is given, we can easily find the local extreme values by **inspection.**



Geogebra file (the journey of tangent line):

https://www.geogebra.org/m/BnfjWrB8

This graph suggests that local maxima or minima occur at the points where the tangent line is horizontal or where the function is not differentiable and this is true.

Fact: Suppose that the function f is defined on an open interval containing the number c. If f has a local minimum or maximum at $x = c$, then

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist.}$$

This fact gives us one of the tools for finding local extreme values of a function defined by a formula.

Definition: The interior points c of the domain of a function f for which

$$f'(c)=0 \quad \text{or} \quad f'(c) \text{ does not exist}$$

undefined

are called **critical points for f.**

A critical point may or may not be a local extreme point; that should be studied after identifying critical points.

Example 1: Find the critical points for the function: $f(x)=x^2-4x-3$.

f': undefined? Never

$f'(x) = 2x-4 = 0$

$\boxed{x=2}$

critical number

Example 2: Find the critical points for the function: $f(x)=x^{1/3}+1$.

Domain $(-\infty,\infty)$

$$f'(x) = \frac{1}{3}\cdot x^{-2/3} = \frac{1}{3\cdot x^{2/3}}$$

$f'=0$? Never

f': undefined? $x=0$ in the domain of f? ✓

critical number: $x=0$.

critical point: $(0, f(0)) = (0,1)$

Finding extreme values

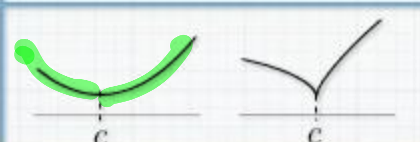
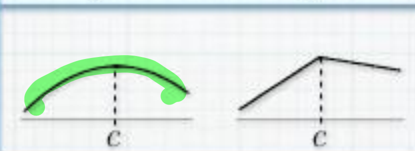
Theorem: The First-Derivative Test

Suppose that c is a critical point for f and f is continuous at c. If there is a positive number r such that:

(i) $f'(x)<0$ for x in $(c-r,c)$ and $f'(x)>0$ for x in $(c,c+r)$, then $f(c)$ is a local minimum.

(ii) $f'(x)>0$ for x in $(c-r,c)$ and $f'(x)<0$ for x in $(c,c+r)$, then $f(c)$ is a local maximum.

(iii) $f'(x)$ has the same sign for all x in $(c-r,c)$ or $(c,c+r)$, then $f(c)$ is not a local extreme value for f.

sign change in $f'(x)$ at c	picture	conclusion
$-\mid+$		local minimum
$+\mid-$		local maximum
none ($-\mid-$ or $+\mid+$)		not a local extremum

f': undef? Never

min or max?

Polynomial

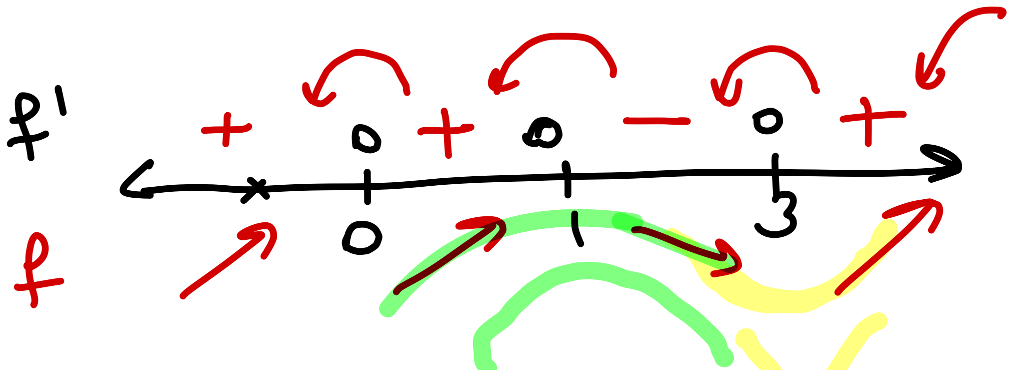
Domain: $(-\infty, \infty)$

Example 3: Locate and classify all local extrema: $f(x) = x^5 - 5x^4 + 5x^3 + 2$.

Given: $f'(x) = 5x^4 - 20x^3 + 15x^2 = \boxed{5x^2(x-3)(x-1)} = 0$

$f'(x) = 0$?

c.p: $\boxed{x = 0, \quad x = 3, \quad x = 1}$



$x = 0$; c.p. Not a local extreme.

$x = 1$; c.p. local max at $x = 1$.

local max point: $(1, f(1))$

local max value: $f(1)$

$x = 3$; a c.p. local min at $x = 3$

local min point: $(3, f(3))$

local min value: $f(3)$ (y-value)

Rational function

Example 4: Find the critical points and local extreme points: $f(x)=\dfrac{x^2}{x-4}$.

Given: $f'(x)=\boxed{\dfrac{x(x-8)}{(x-4)^2}}$

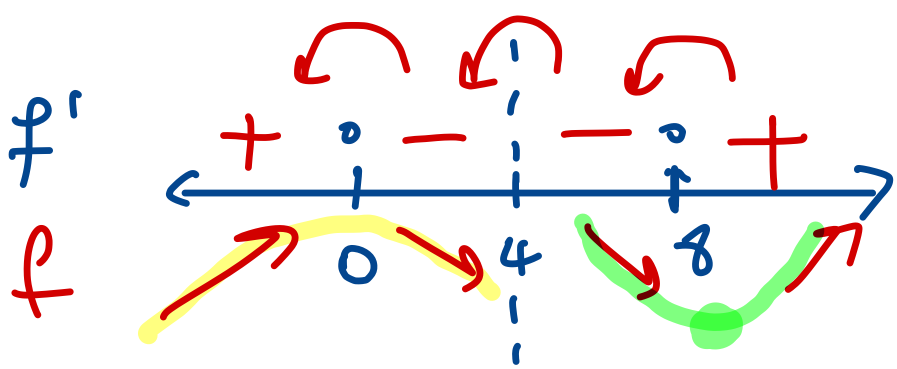
Domain: $x \neq 4$.
VA: $x=4$

$f'(x)=0$ if top $=0 \Rightarrow \underbrace{x=0}_{\text{c.p.}}, \ \underbrace{x=8}_{\text{c.p.}}$

f': undefined if bottom $=0 \Rightarrow \boxed{x=4}$.

must be used on the sign chart

Not a critical point since not in the domain of $f(x)$.



critical points: $x=0$ & $x=8$

classify

local max at $x=0$

$(0, f(0))$

local min at $x=8$

$(8, f(8))$

f increases on

$(-\infty, 0)$ & $(8, \infty)$

f decreases on

$(0, 4)$ & $(4, 8)$

Example 5: Find the critical points. Determine the intervals where the function is increasing/decreasing. Locate all local extrema.

$$\boxed{f(x)=\frac{x^2+1}{x^2-1}.}$$

Domain: $x \neq 1$, $x \neq -1$

VA: $x = 1$, $x = -1$

Given: $f'(x)=\frac{-4x}{\left(x^2-1\right)^2}=\frac{-4x}{(x-1)^2(x+1)^2}$

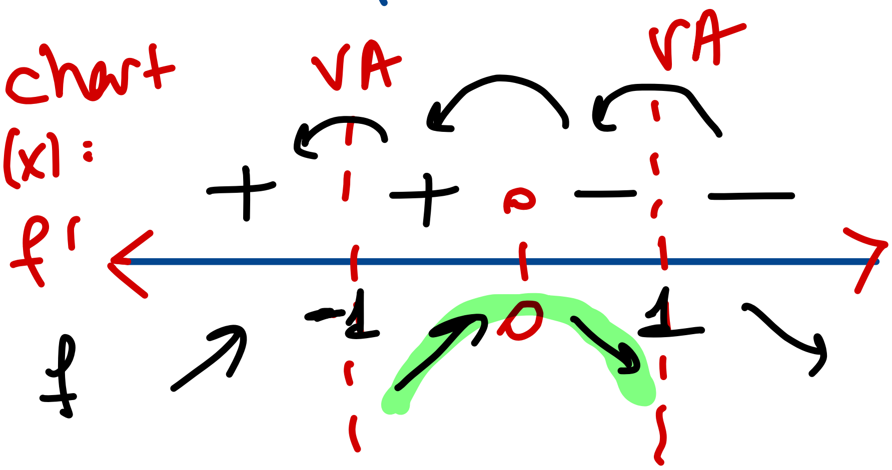
$f'(x) = 0$ if top $= 0$ $\Rightarrow$ $x = 0$

$f'(x)$: undef. if bottom $= 0$ $\Rightarrow$ $\underbrace{x = 1,\ x = -1}$

Not in domain of f.

critical point: $\boxed{x = 0}$

Sign chart for $f'(x)$:



f is increasing on: $(-\infty, -1)$ & $(-1, 0)$

f is decreasing on: $(0, 1)$ & $(1, \infty)$

critical point: $x = 0$

local max at $x=0$

local max value : $f(0)$

$$f(0) = \frac{0+1}{0-1} = -1$$

$(0, -1)$ ← local max point

Radical function

$\sqrt[3]{(x-5)^2}$ Domain: $(-\infty,\infty)$

Example 6: Find the critical points and classify them as local min/max or neither: $f(x)=(x-5)^{2/3}$.

$$f'(x) = \frac{2}{3}(x-5)^{-1/3} = \frac{2}{3(x-5)^{1/3}}$$

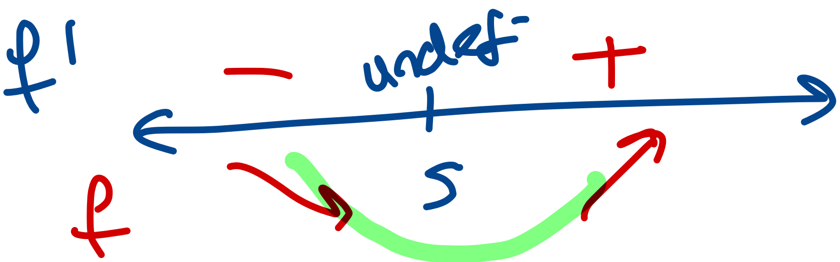
$\sqrt[3]{(x-5)}$

$f'(x)=0$? Never.

$f'(x)=$ undef.? bottom $=0$? $x=5$

Is $x=5$ a critical point? ✓

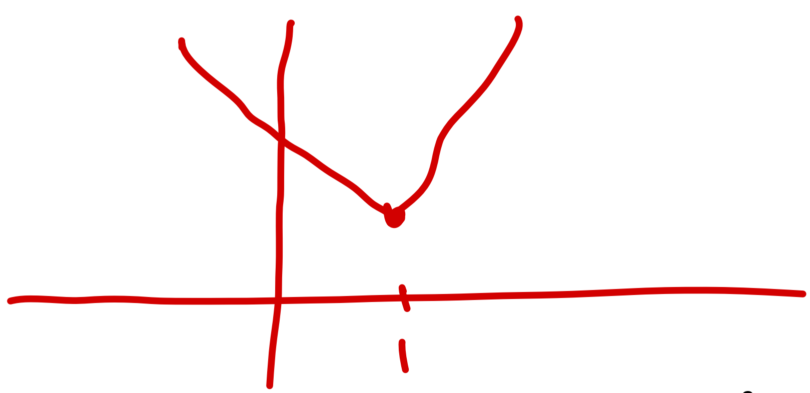
(in the domain of $f(x)$).



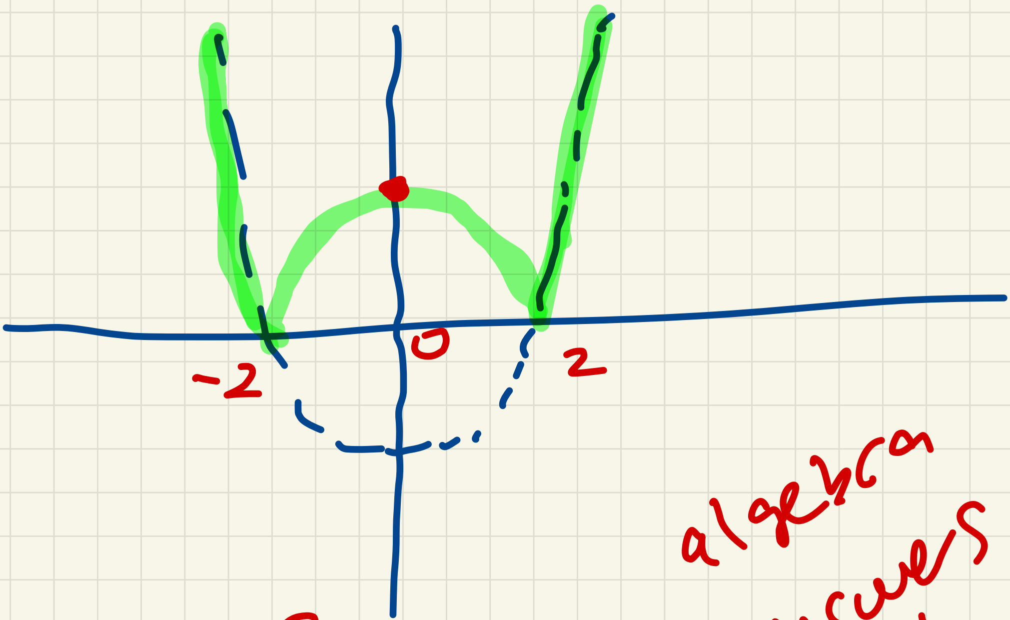
local min at $x=5$.

Note that, if you know the graph of the function, that might be helpful to find critical points:

Exercise: Find the critical points for the function: $f(x)=2|x-1|+4$.



$$f(x) = |x^2 - 4|$$



c.p?

algebra answers it.

Remark: Sometimes it is difficult to study the sign of the derivative function. For some cases, it may be easier to use the following test:

Theorem: The Second-Derivative Test

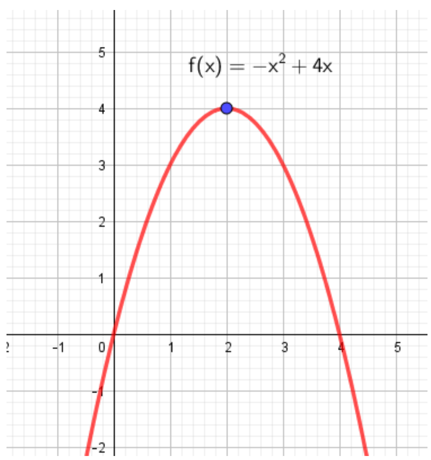
Let c be a critical point for f where $f'(c)=0$ and $f''(c)$ exists.

(i) If $f''(c)>0$, then $f(c)$ is a local minimum value, +

(ii) If $f''(c)<0$, then $f(c)$ is a local maximum value, −

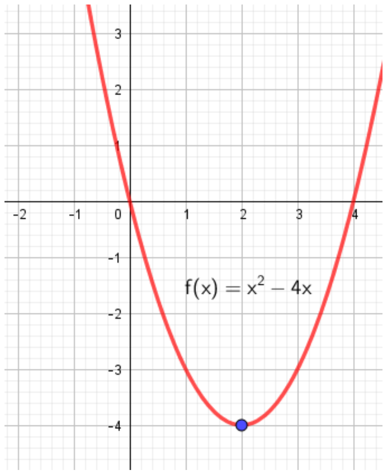
(iii) If $f''(c)=0$, then this test is inconclusive.

Example:



Observe: $f(x)=-x^2+4x$; $f'(x)=-2x+4$; $f''(x)=-2$

$x=2$ is a critical point with $f''(2)<0$; the second derivative test concludes $x=2$ should be a local maximum. The graph (or the first derivative test) confirms this result.



Observe: $f(x)=x^2-4x$; $f'(x)=2x-4$; $f''(x)=2$

$x=2$ is a critical point with $f''(2)>0$; the second derivative test concludes $x=2$ should be a local minimum. The graph (or the first derivative test) confirms this result.

That is, when needed, we can use the second derivative test to check whether a critical point is a local minimum or maximum.

If you are only interested in a behavior of a single critical point, using the second derivative test saves time:

Example 7: Consider the function $f\left(x\right)=3x^4-8x^3-6x^2+24x+5$. We know that $x=1$ is a critical point for this function. Determine whether this critical point is a local minimum or maximum for the function.

$f'(x) = 12x^3 - 24x^2 - 12x + 24$

$f''(x) = 36x^2 - 48x - 12$

$x=1$ CP ; $f''(1) = 36 - 48 - 12$

$f''(1) = -24 < 0$

2nd derivative test says: $x=1$ is a $\boxed{\text{local max}}$

For instance, with trig functions, it might be difficult to study the sign of the first derivative. Using second derivative test helps.

Example 8: Given: $f(x) = \sin(x) + \cos(x)$ over the interval $[0, 2\pi]$. Find all critical points and classify them as local min/max over this interval.

Solution:

Find the first and second derivatives:

$f'(x) = \cos(x) - \sin(x)$

$f''(x) = -\sin(x) - \cos(x)$

Find the critical points. Critical points are where the derivative is 0 or undefined:

Solve: $f'(x) = \cos(x) - \sin(x) = 0$

Use the second derivative test on the critical points:

Exercise: Locate local extrema for $f(x) = x + \cos(2x)$, over $[0, \pi]$.

If you are only given the second derivative of a function and some critical points, then you must use the second derivative test as first derivative is not available (we don't know what the first derivative is):

Example 9: Let $f(x)$ be a function with critical points $x=1$ and $x=4$. If $f''(x)=6x-12$, classify the critical points of $f(x)$ as local min/max.

2^{nd} derivative test.

$x=1 \Rightarrow f''(1) = 6\cdot 1-12 = -6 < 0$

local max at $\underline{\underline{x=1}}$.

$x=4 \Rightarrow f''(4) = 6\cdot 4-12 = 12 > 0$

local min at $\underline{\underline{x=4}}$

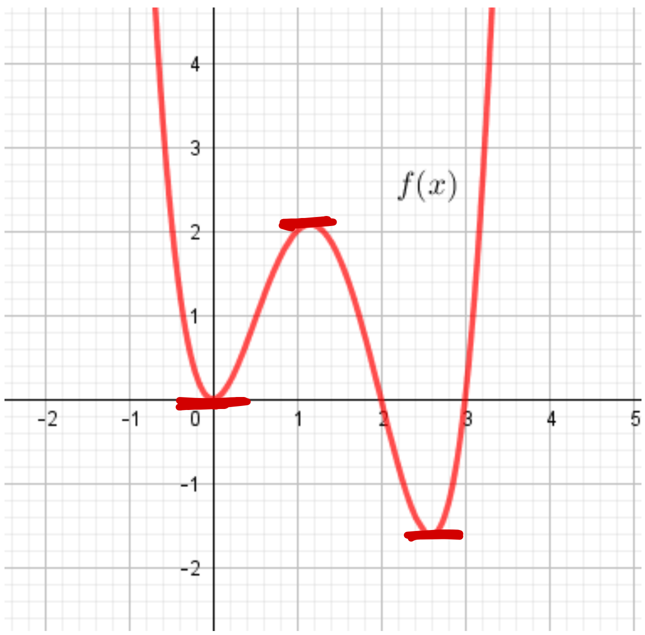
Exercise: Given $f(x)=\dfrac{x^2-1}{x}$, find the critical points, and use the second derivative test to classify them.

Graphs of f and f'

Given the graph of f , what can you figure out about f'?

We can observe the intervals where the function is increasing and decreasing; that is, we can see when $f'>0$ or $f'<0$. We can observe the critical points; and points where the tangent line is horizontal (that is, the roots of f').

Example: Given the graph of $f(x)$, answer the following:



The critical points of $f(x)$ are: _____________

$f'(x)>0$ over the interval(s): ______________

The roots of $f'(x)$ are: ____________________

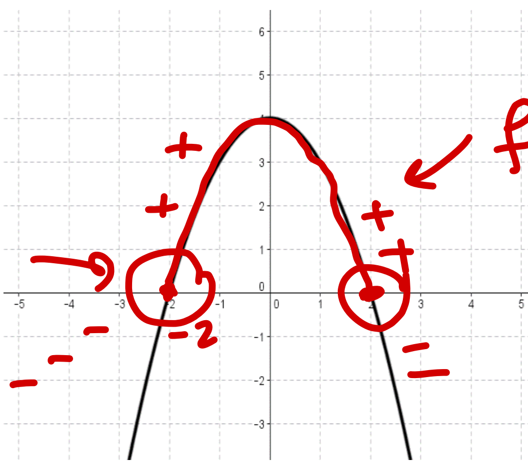
Question: Given the graph of f', what can you figure out about f?

The intervals where $f' > 0$ would give us when f(x) is increasing.

The roots of $f'(x)$ would give us "some" of the critical points of f(x).

The sign change around the roots of $f'(x)$ would give us information about local min/max.

Example: Given the graph of f', answer the following:



f' − 0 + 0 −

−2 2

when is f' > 0

The function f is increasing over: (-2, 2)

Critical points of $f(x)$ are: x = −2, x = 2

$f(x)$ has local min at: x = -2, local max at: x = 2

cp

f' − +

cp

f' + −

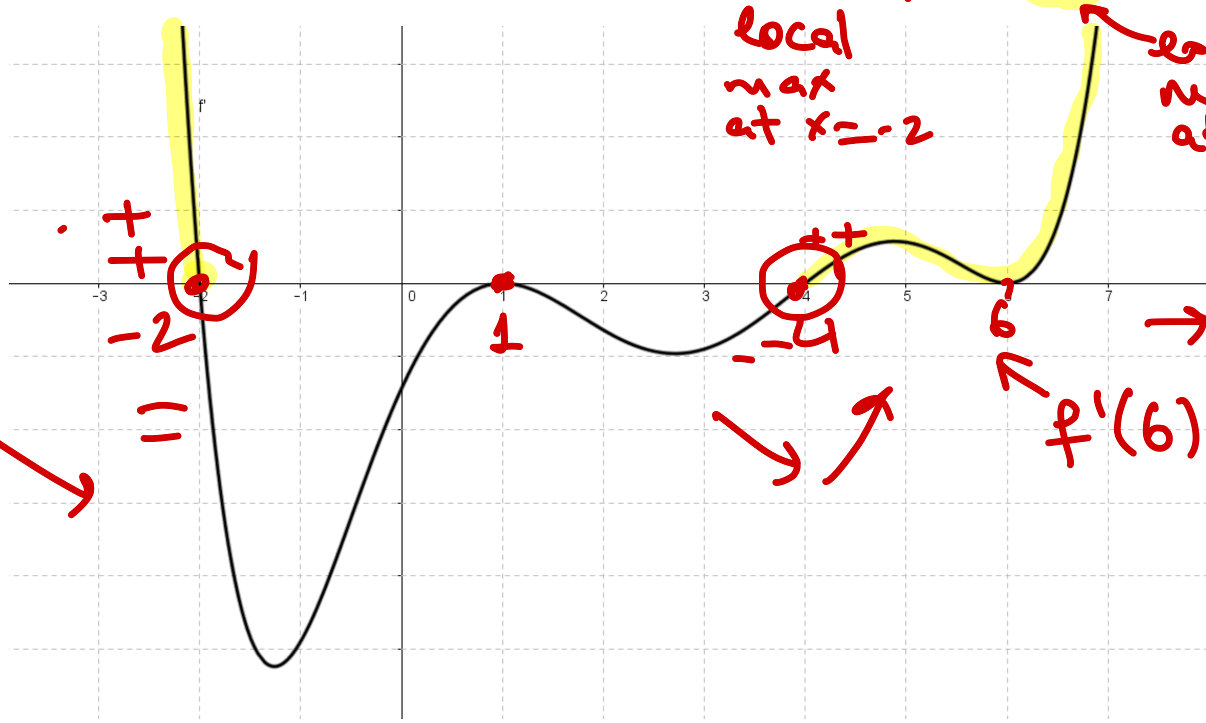
Example: The graph of f' is given below.

f': $+ \; 0 \; - \; 0 \; - \; 0 \; + \; 0 \; +$

$\nearrow \; -2 \; \searrow \; \boxed{1} \; \searrow \; 4 \; \nearrow \; \boxed{6} \; \nearrow$

local max at $x=-2$

local min at $x=4$



$f'(6)=0$

When is f increasing? $\Rightarrow$ if $f'>0$

$(-\infty,-2)$ & $(4,6)$ & $(6,\infty)$

can combine: $(4,\infty)$

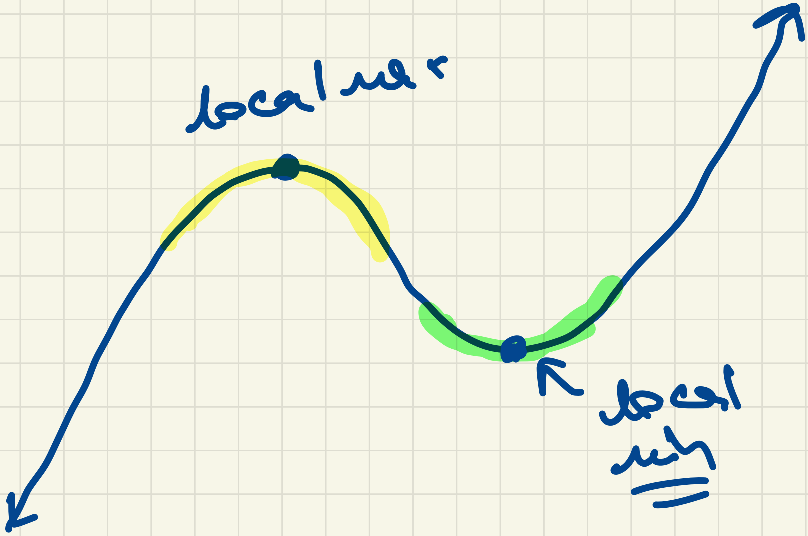
What are the critical points of f?

CP: $-2, 1, 4, 6$

Locate the x-coordinates of the local extreme values of f?

local min : $x=4$

local max : $x=-2$



$f(x)$ over $[-2, 5]$

Section 3.4 Continued:

Absolute Extreme Values

Definition: Let c be a point in the domain of f; c may be an interior point or an endpoint.

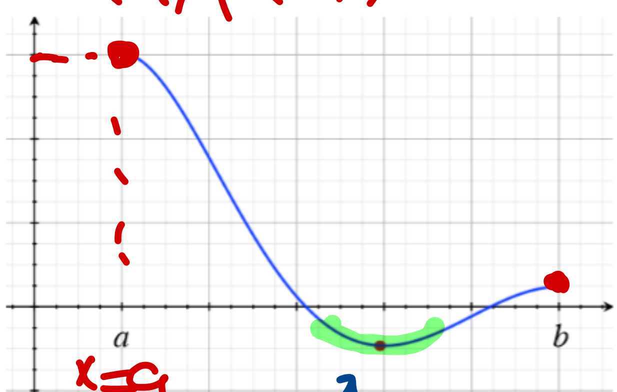
We say that

f has an **absolute minimum** at c if $f(x) \geq f(c)$ for all x in the domain of f,

f has an **absolute maximum** at c if $f(x) \leq f(c)$ for all x in the domain of f.

1

(a, f(a))



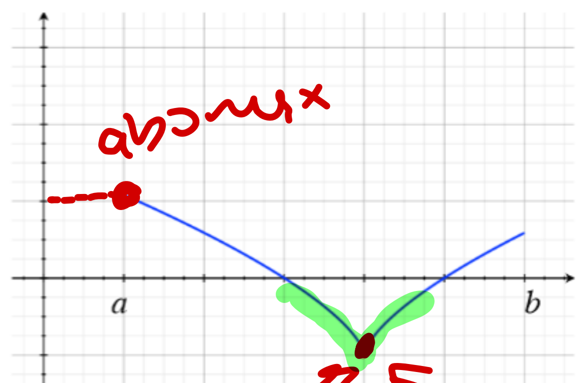
x=a

abs max at x=a.

local min & abs min

2

abs max



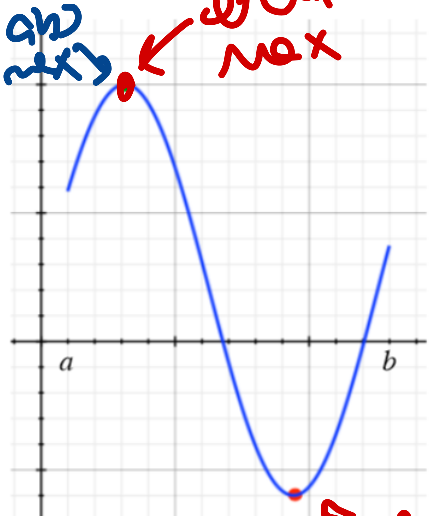
abs min

local min.

abs max

local max

4

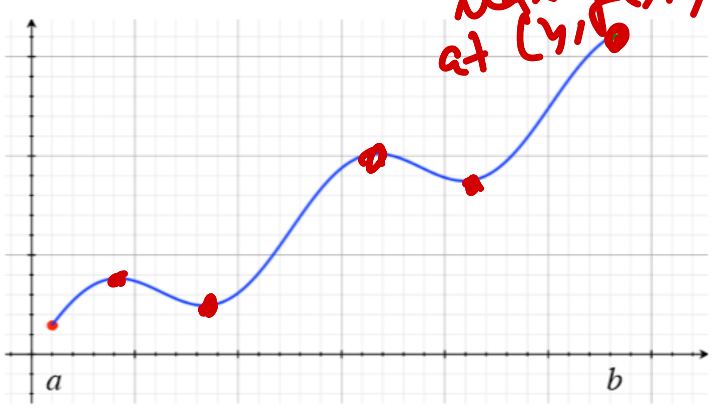


abs min

local min

abs max at (b, f(b))

3



abs min at (a, f(a))

NOTE: If f takes on an absolute extreme value, then it does so at a critical point or at an endpoint.

Finding the absolute minimum and maximum values of a <u>continuous</u> function defined on a closed bounded interval $[a,b]$:

1. Find the critical points for f in the interval (a,b).
2. Evaluate the function at each of these critical points and at the endpoints.
3. The smallest of these computed values is the absolute minimum value, and the largest is the absolute maximum value of f.

absolute!

Example: Find and classify all extreme values of the function

$$\boxed{f(x) = x^3 + 3x^2 - 1}\ [-3,2].$$

endpoints: -3 & 2

Solution:

Step 1: Find the critical points (that lie in this interval).

$f'(x) = 3x^2 + 6x = 0$

$\Rightarrow\quad 3x(x+2) = 0$

$\boxed{x = 0,\quad x = -2}$

f': $+\quad 0\quad -\quad 0\quad +$

$\boxed{-2}\qquad \boxed{0}$

local max at $(-2, f(-2))$

local min at $(0, f(0))$

Step 2: Make a chart with function values:

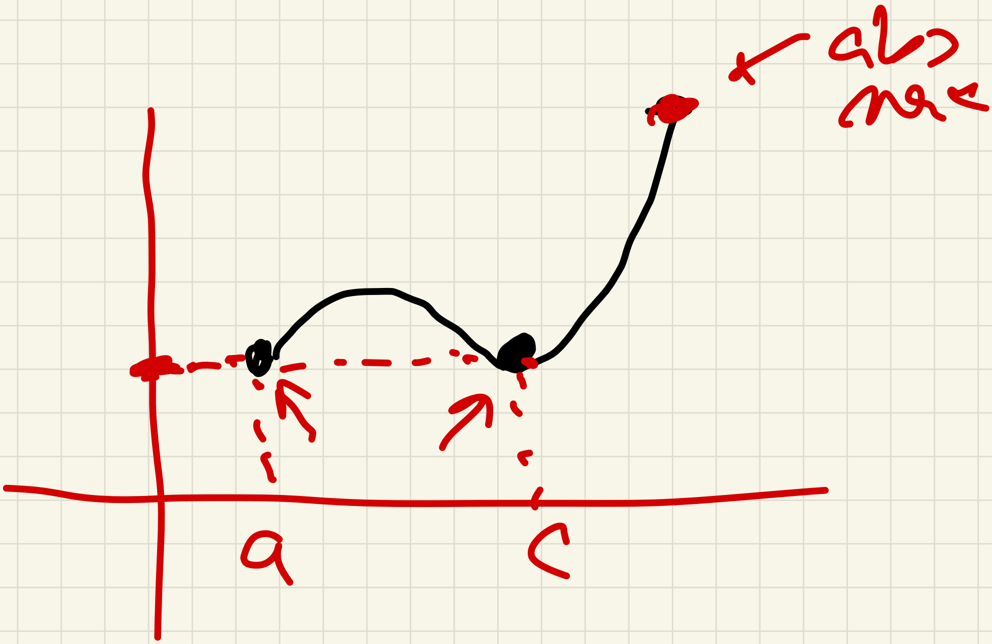
	x	$f(x)$
endpt	-3	$f(-3) = -1$
c.p	-2	$f(-2) = 3$
c.p	0	$f(0) = -1$
endpt	2	$f(2) = 2^3 + 3 \cdot 2^2 - 1 = 19$

abs max at $(2, 19)$

absolute max. value: $\underline{\underline{19}}$.

absolute min at

$(-3,-1)$ & $(0,-1)$



Exercise: Find and classify all absolute extreme values of the function

$$f(x) = x^2 - 4x + 1 \ , \ [0,3].$$

Exercise: Find and classify all absolute extreme values of the function

$$f(x) = \tan(x) - x \ , \ \left[-\frac{\pi}{4}, \frac{\pi}{2}\right).$$

Hint: $f'(x) = \sec^2(x) - 1 = 0 \rightarrow \sec(x) = \pm 1$