completed

Math 2413- Calculus I

Dr. Melahat Almus

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- Check CASA calendar for due dates.
- Bring “blank notes” to class. Completed notes will be posted after class.
- Do your best to attend every lecture and lab.
- Study after every lecture; work on the quiz covering the topic we cover on the lecture immediately afterwards. Retake your quizzes for more practice.
- Get help when you need help; bring your questions to the labs, or my office hours. We also have tutoring options on campus.
- Respect your friends in class; stay away from distractive behavior. Do your best to concentrate on the lecture.
- If you email me, mention the course code in the subject line. Email is the best way to communicate with me outside of class. Teams chat messages are not monitored or replied to.

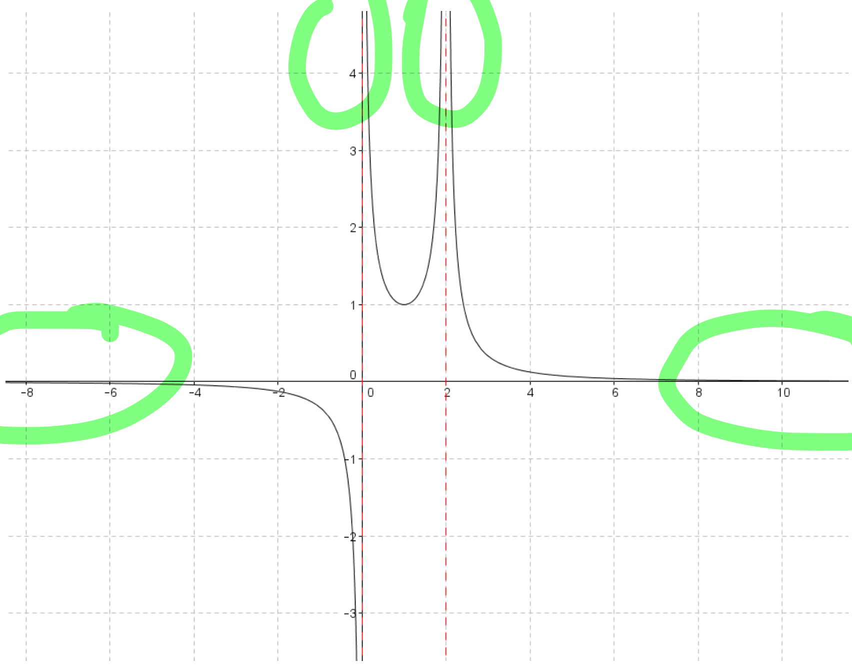
Section 3.6 – Curve Sketching

Recall--- I assume you know how to find horizontal and vertical asymptotes. Read the next pages and solve the examples before lecture!

Vertical and Horizontal Asymptotes

If $f(x) \to \pm\infty$ as $x \to c^{+}$ or $x \to c^{-}$, then the line $x = c$ is a **vertical asymptote** for $f(x)$.

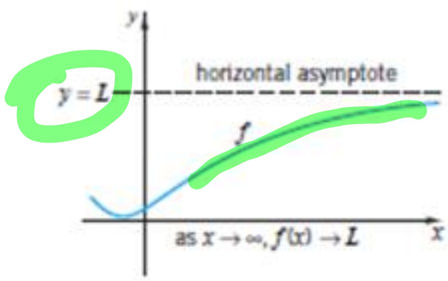
The graph of $f(x) = \dfrac{1}{x|x-2|}$ is given below.

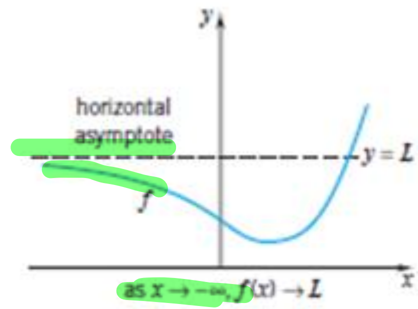


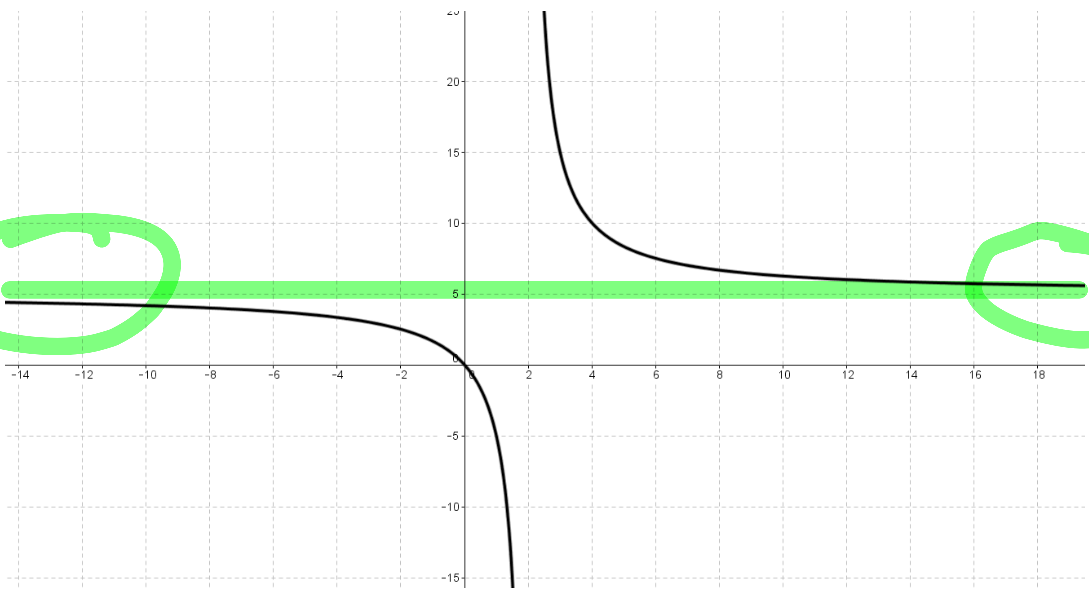
As we saw in Section 1.3, the behavior of a function as $x \to \pm\infty$ determines the **horizontal asymptotes**.

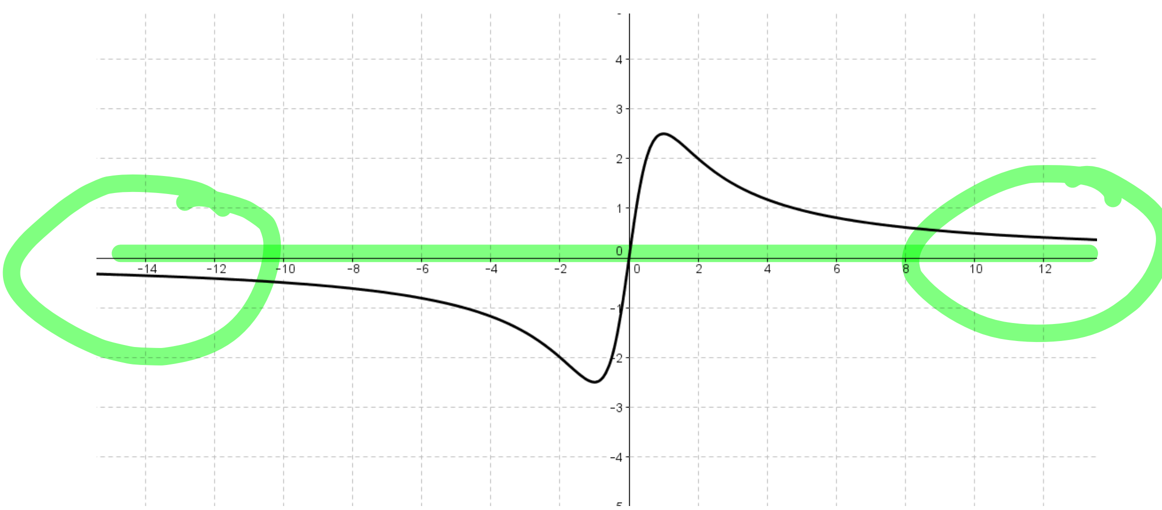
If $\lim\limits_{x\to\infty} f(x) = L$, then the line $y = L$ is a (rightward) horizontal asymptote.

If $\lim\limits_{x\to-\infty} f(x) = L$, then the line $y = L$ is a (leftward) horizontal asymptote.


horizontal asymptote
y = L
f
x
y
as x → ∞, f(x) → L

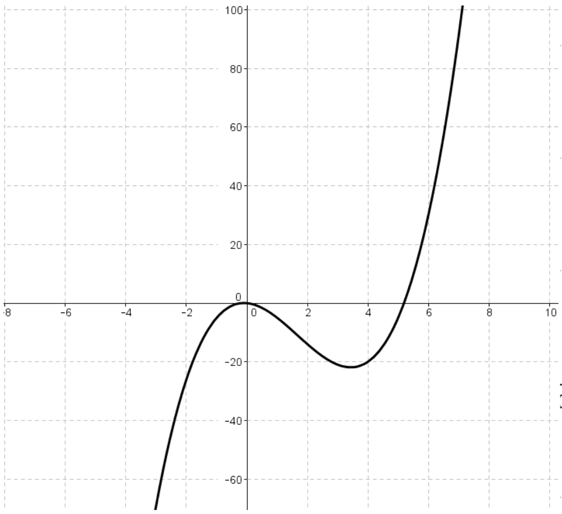

horizontal asymptote
y = L
f
x
y
as x → −∞, f(x) → L

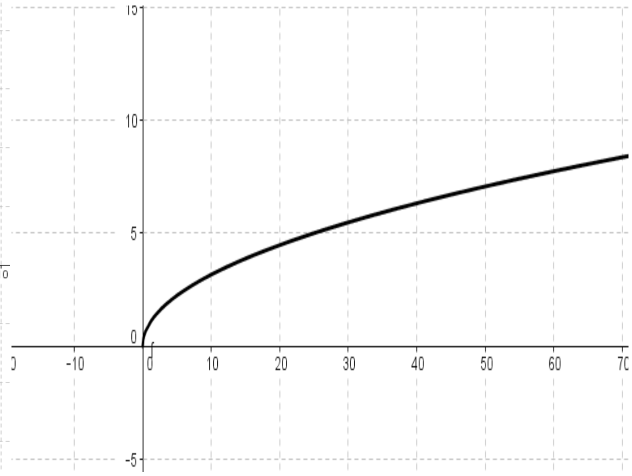




HA
$y=0$

Note that some functions don't have any vertical or horizontal asymptotes; this can be seen on the graphs given below.





Given the formula, you should be able to find the vertical and horizontal asymptotes of a function.

$$f(x)=\frac{x+1}{x^2-4}$$

$$f(x)=\frac{x+1}{2+5x}$$

$$f(x)=\frac{x^2}{4+\cos x}$$

$$f(x)=\frac{x^2}{1-2\sin x}$$

$\sin(x)=\frac{1}{2}$

$x=\frac{\pi}{6},\ \frac{5\pi}{6},\ \ldots$

$$f(x)=x^5-4x^3+1$$

Vertical and Horizontal Asymptotes

If $f(x) \to \pm\infty$ as $x \to c^+$ or $x \to c^-$, then the line $x = c$ is a **vertical asymptote** for $f(x)$.

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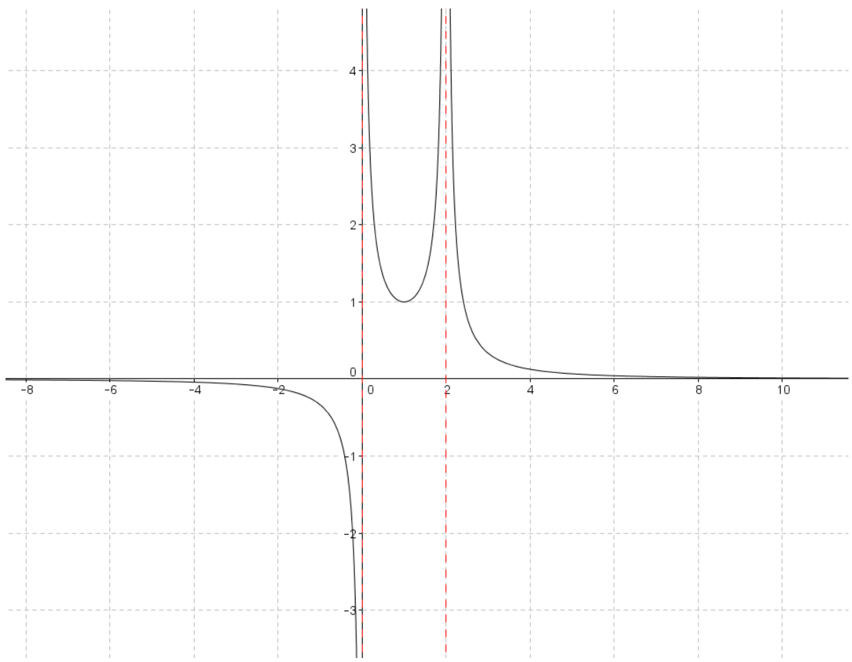

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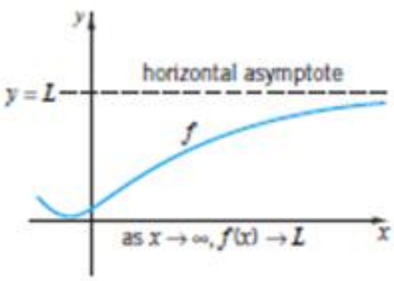

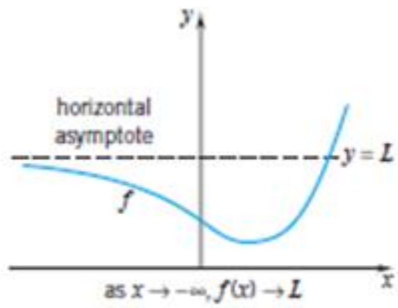

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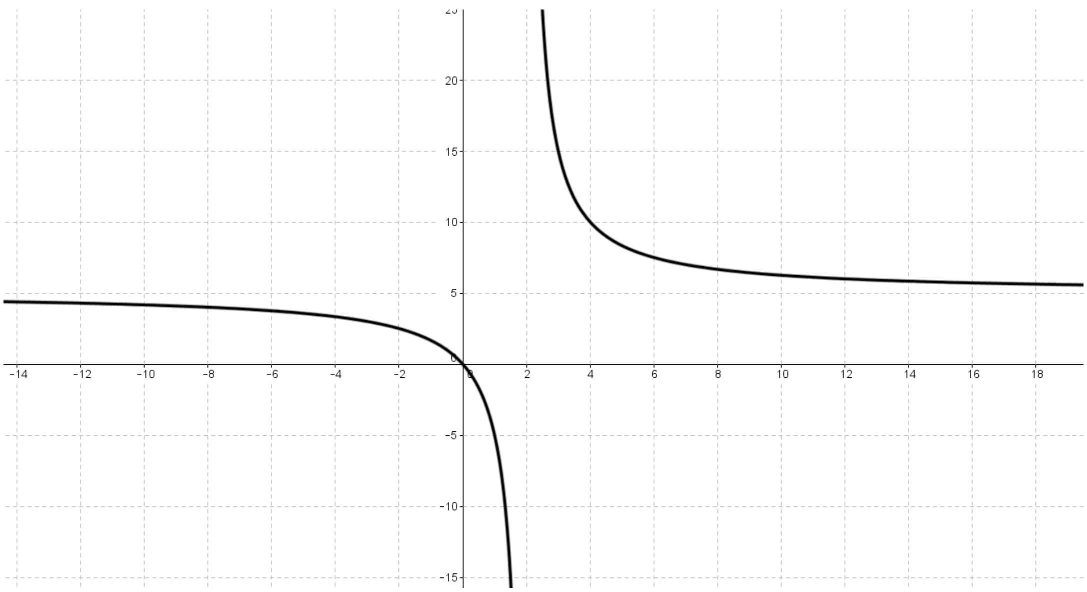
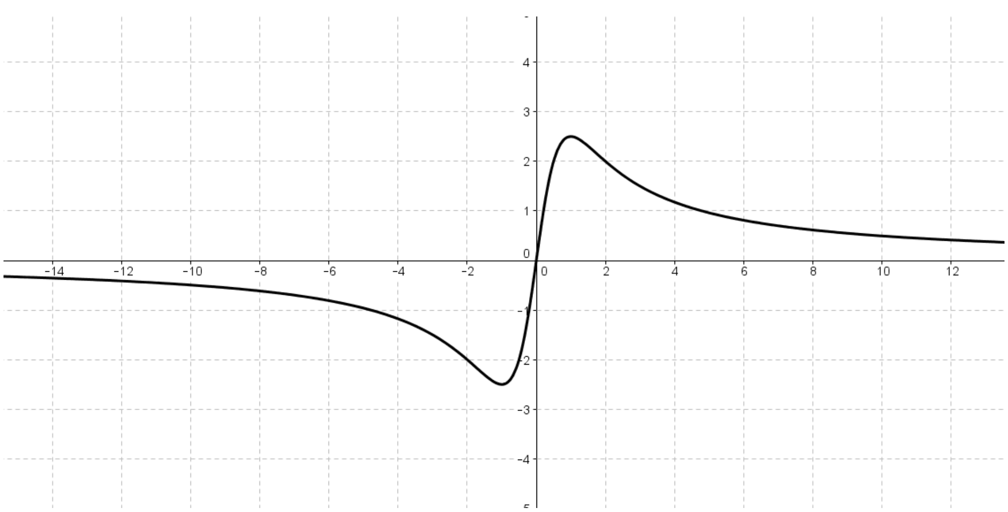
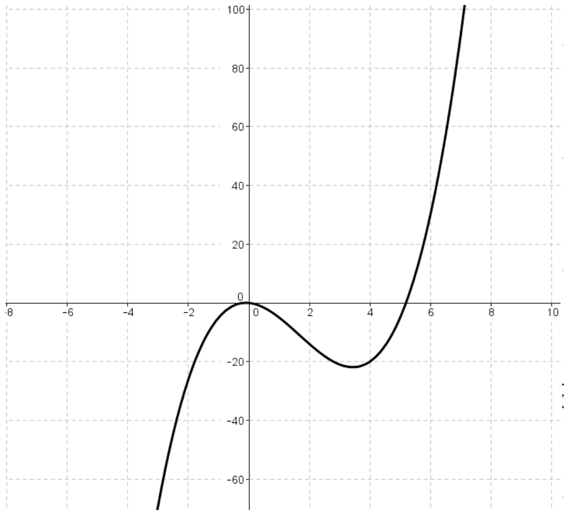
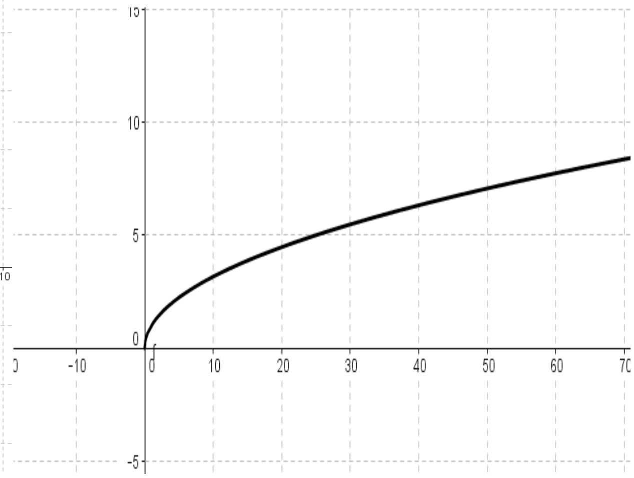








NOTE: Polynomials do not have any asymptotes.

Given the formula, you should be able to find the vertical and horizontal asymptotes of a function.

$$f(x) = \frac{x+1}{x^2-4}$$

$$f(x) = \frac{x+1}{2+6x}$$

$$f(x) = \frac{x^2}{4+cos(x)}$$

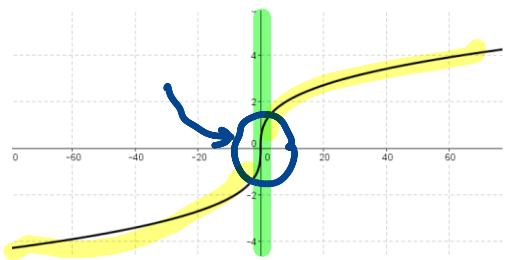
$$f(x) = \frac{x^2}{1-2sin(x)}$$

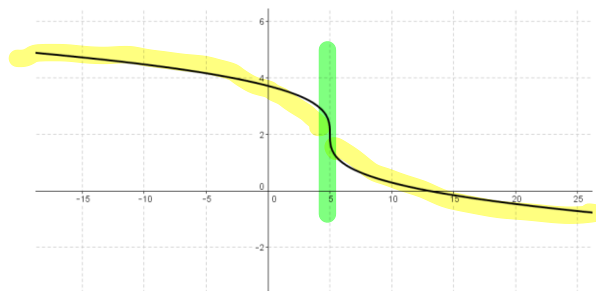
$$f(x) = x^5 - 4x^3 + 1$$

NEW: Vertical Tangents

Suppose that $f(x)$ is continuous at $x = c$.

If $f'(x) \to \infty$ or $f'(x) \to -\infty$ as $x \to c$, then we say that the function has a **vertical tangent** at the point $(c, f(c))$.





Some radical functions have vertical tangents. You will observe that"

- $f(c)$ is defined
- $f'(c)$ is undefined
- The sign chart for $f'(x)$ across $x = c$ has no sign change.

Radical fn.

$$\frac{1}{3\cdot\left(\sqrt[3]{(x-9)}\right)^2}$$

Example: $f(x)=(x-9)^{1/3}$. Classify all critical points.

Domain: $\mathbb{R}$

Solution:

$f'=0$? X

f': undefined? cp $\boxed{x=9}$

$f'(x)=\dfrac{1}{3(x-9)^{2/3}}$; the critical point is:

Sign chart for $f'(x)$:

f': + undefined +

f: ↗ $\boxed{9}$ ↗

cp

Hence, the function has a vertical tangent at x=9.

no sign change

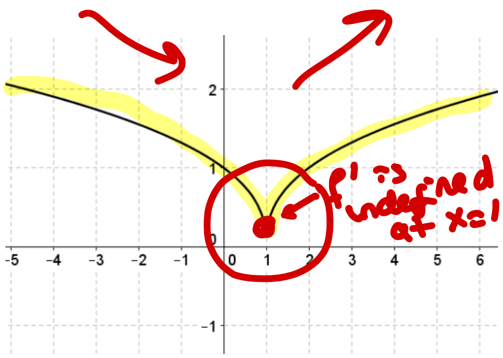
$\Rightarrow$ V.T.

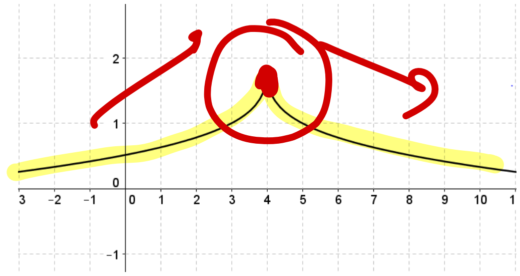
$$(\ \ )^{2/3}=\left((\ \ )^{1/3}\right)^{\boxed{2}}$$

Vertical Cusps

Suppose that $f(x)$ is continuous at $x = c$.

If $f'(x) \to \infty$ as $x \to c$ from one side and $f'(x) \to -\infty$ as $x \to c$ from the other side, then we say that the function has a **vertical cusp** at the point $(c, f(c))$.





Some radical functions have vertical ~~tangents~~ cusp. You will observe that”

- $f(c)$ is defined
- $f'(c)$ is undefined
- The sign chart for $f'(x)$ across $x = c$ has sign change.

$((\quad)^{1/7})^5$

Radical

Domain: $\mathbb{R}$

Example: $f(x) = 2(x-4)^{2/7}$. Classify all critical points.

Solution:

$f'(x) = \dfrac{\boxed{8}}{7(x-4)^{5/7}}$; the critical point is:

$f' = 0$? ✗ c.p.

f': undefined? $\boxed{x=4}$

Sign chart for $f'(x)$:

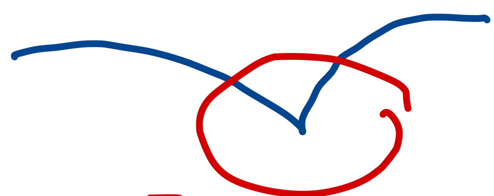
f': $-$ | undef. | $+$

f: ↘ $\boxed{4}$ ↗

Hence, the function has ______________ at_______.

vertical cusp at $x=4$.

(sign change)



Exercise: $f(x) = (x^2-4)^{1/5}$. Classify all critical points.

$f'(x) = \frac{1}{5} \cdot (x^2-4)^{-4/5} \cdot (2x)$

$f'(x) = \dfrac{2x}{5(x^2-4)^{4/5}}$

$f' = 0$? cp $\boxed{x=0}$

f': undef? cp $\boxed{x=2, x=-2}$

f': $-$ undef $-$ 0 $+$ undef $+$

↘ $\boxed{-2}$ ↘ 0 ↗ $\boxed{2}$ ↗

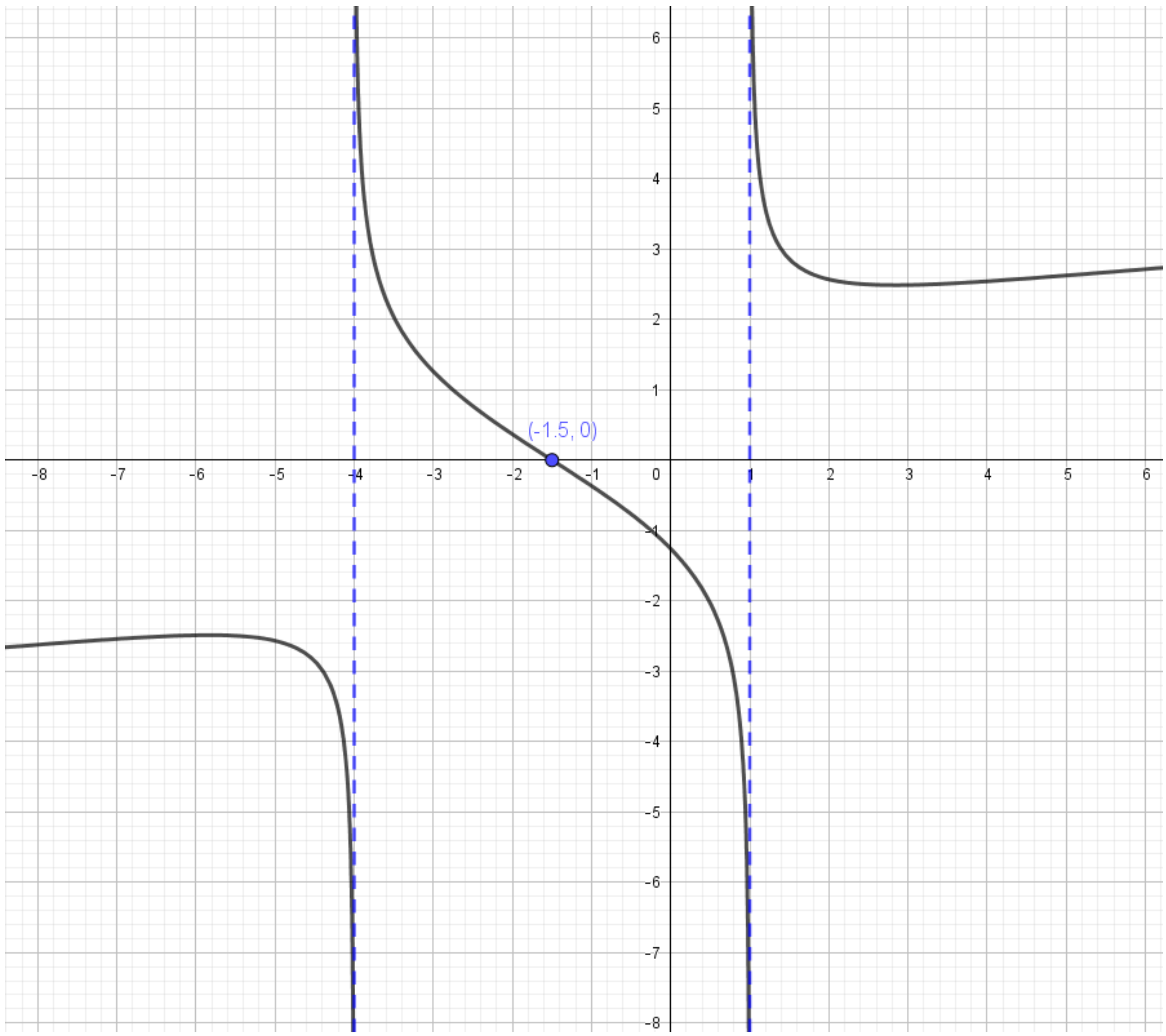
no sign change

V.T. at x = -2

x=0 is a local min.

V.T. at x=2.

Exercise: The function $f(x)$ is defined on all real numbers. The graph of its first derivative is given below. $f'(x)$ is shown below.

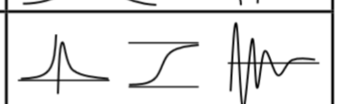
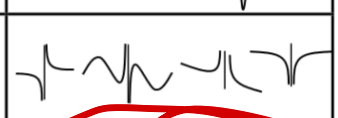
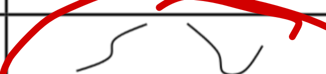
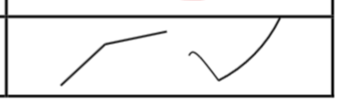


What are the critical points of f(x)? Classify them.

Using Calculus to graph a function.

1. Determine the **domain** of the function f. Find any vertical asymptotes and study the behavior of f as $x \to \pm\infty$.
2. Determine any **intercepts** of the function. To find the $x-$intercepts, we need to solve the equation $f(x)=0$ and to find the $y-$intercepts, evaluate the function at 0 (if 0 is in the domain of f).
3. Find the **first derivative,** f'. Determine any critical points, intervals of increase/decrease, local extreme points, vertical tangents and cusps.
4. Find the **second derivative,** f''. Study the sign of f'' to understand concavity of the function and determine any points of inflection.
5. Plot the **points of interest** (intercepts, local or absolute extreme points, points of inflection).
6. Sketch the graph of f using the information gathered in the previous steps. Make sure that the function has the right shape (concaves up/down, rises/falls) on the corresponding intervals.

Summary of Graphical Features

$f'(x)>0$ / $f'(x)<0$	**increasing / decreasing**	
$f''(x)>0$ / $f''(x)<0$	**concave up / down**	
$\lim_{x\to-\infty} f(x)=b$ or $\lim_{x\to\infty} f(x)=b$	**horizontal asymptote at $y = b$**	
$\lim_{x\to a^-} f(x)\to\pm\infty$ or $\lim_{x\to a^+} f(x)\to\pm\infty$	**vertical asymptote at $x = a$**	
$\lim_{x\to a} f'(x)\to\pm\infty$	**vertical tangent at $x = a$**	
$\lim_{x\to a^-} f'(x)\to\pm\infty$ $\lim_{x\to a^+} f'(x)\to\mp\infty$	**cusp at $x = a$**	
$\lim_{x\to a^-} f'(x)\neq \lim_{x\to a^+} f'(x)$	**"corner" at $x = a$**	

$f(x)=0$?

$f(0)=0$

$(0,0)$

y-int.

Example: Graph $f(x)=x(x-1)^{\frac{1}{3}}$. Find the intervals of increase/decrease, concave up/down. Find critical points and classify them.

Solution:

Domain: $\mathbb{R}$

Asymptotes: None

Intercepts: $\boxed{x=0, \quad x=1}$ x-intercepts

$(0,0)$ y-int $(1,0)$

Critical Points (classify them) :

$$f'(x)=\frac{4x-3}{3(x-1)^{\frac{2}{3}}}$$

$f'=0 \Rightarrow \boxed{x=3/4}$ cp

f': undef $\Rightarrow \boxed{x=1}$ cp

Sign Chart for the first derivative:

$f'(x)$: $-$ 0 $+$ undef $+$

f: $\searrow$ $\boxed{3/4}$ $\nearrow$ $\boxed{1}$ $\nearrow$

local min at $\left(\frac{3}{4}, f\left(\frac{3}{4}\right)\right)$

V. Tangent at $x=1$

$(1, f(1)):(1,0)$

Now, second derivative:

$$f''(x)=\frac{4x-6}{9(x-1)^{\frac{5}{3}}}$$

Possible POIs to study:

$f''=0? \Rightarrow x=\frac{6}{4}=\frac{3}{2}$

f'': undef? $\Rightarrow x=1$

Sign Chart for the second derivative:

$f''(x)$: $+$ undef $-$ 0 $+$

f: $\cup$ $\boxed{1}$ $\cap$ $\boxed{3/2}$ $\cup$

PoI at $(1, f(1))$

PoI at $\left(\frac{3}{2}, f\left(\frac{3}{2}\right)\right)$

$f(x) = x(x-1)^{1/3}$

Intervals of Increase:

$f\left(\frac{3}{4}\right) = \underbrace{\frac{3}{4}\cdot\left(-\frac{1}{4}\right)^{1/3}}_{} < 0$

Intervals of Decrease:

Concave up on:

$f\left(\frac{3}{2}\right) = \frac{3}{2}\cdot\left(\frac{1}{2}\right)^{1/3} > 0$

Concave down on:

$f(x)=0$ $\quad f(-1/2) = +4\cdot\frac{1}{8} - 6\cdot\frac{1}{4} + 24\cdot\frac{-1}{2} + 12$

Example: Find the domain, critical points, relative extrema, intervals of increase decrease, concavity, and graph $\quad f(x) = -4x^3 - 6x^2 + 24x + 12$

$f(-2) = 32 - 24 - 48 + 12 = -28$

$f(1) = -4 - 6 + 24 + 12 = 26$

Solution:

Domain: $(-\infty, \infty)$

Asymptotes: None.

Intercepts: y-int: $f(0) = 12 \quad (0, 12)$

Critical Points (classify them):

$f'(x) = -12x^2 - 12x + 24 = -12(x^2 + x - 2) = -12(x+2)(x-1)$

cp: $x = -2$, $x = 1$

Sign Chart for the first derivative:

$f'(x)$: $-$ $\quad 0$ at -2 $\quad +$ $\quad 0$ at 1 $\quad -$

$\searrow \quad -2 \quad \nearrow \quad 1 \quad \searrow$

local min at $(-2, f(-2))$

local max at $(1, f(1))$

$f''(x) = -24x - 12 = -12(2x+1) \qquad f''(x) = 0 \Rightarrow x = -1/2$

Sign Chart for the second derivative:

$f''(x)$: $+$ $\quad 0$ at $-1/2$ $\quad -$

f: $\cup \quad -1/2 \quad \cap$

PoI at $\left(-\frac{1}{2}, f\left(-\frac{1}{2}\right)\right)$

Intervals of Increase:

$(-2, 1)$ $\qquad \left(-\frac{1}{2}, \ -1\right)$

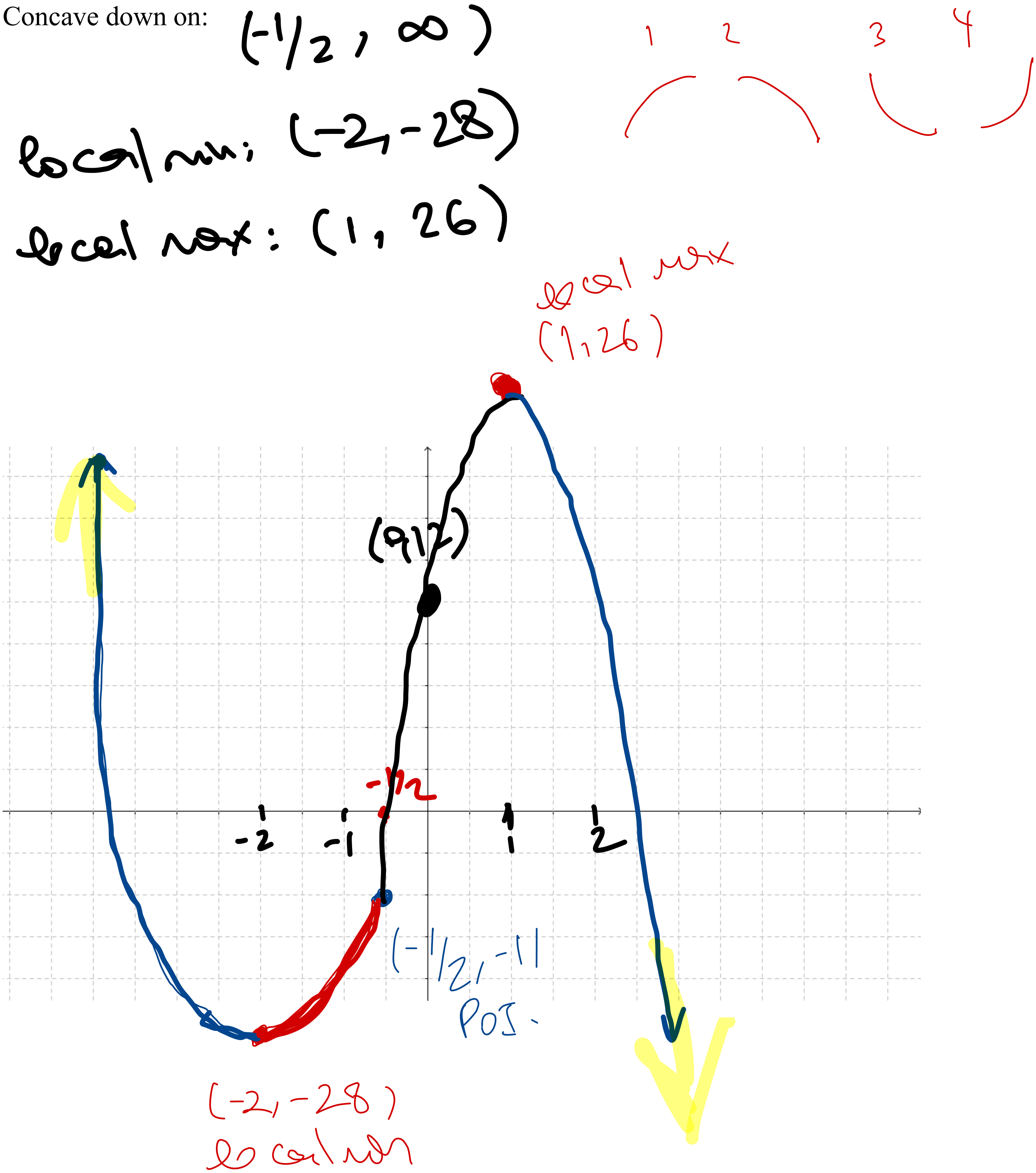
Intervals of Decrease: $(-\infty, -2)$ & $(-2, 1)$

Concave up on: $(-\infty, -1/2)$

Concave down on: $(-1/2, \infty)$

local min: $(-2, -28)$

local max: $(1, 26)$



Day 16 : We'll finish 3.6. WHW7: 3.5 & 3.6
Main review is posted on CASA.

Example: Graph $f(x) = \dfrac{2x}{x^2+1}$. Find the intervals of increase/decrease, concave up/down. Find critical points and classify them. Find and label any points of inflection.

$x^2+1 \neq 0$

$f(-1) = \frac{-2}{2} = -1 \qquad f(1) = \frac{2}{2} = 1$

Solution:

Domain: $(-\infty, \infty)$

Asymptotes: VA: none HA: $y=0$

Intercepts: $(0,0)$

Critical Points (classify them):

$$\boxed{f'(x) = \frac{-2x^2+2}{(x^2+1)^2} = \frac{-2(x^2-1)}{\boxed{(x^2+1)^2}}}$$

$-$, $(x-1)\cdot(x+1)$: $+$, $+$; denominator: $+$

$f'=0$? cp $\boxed{x = \pm 1}$

f' undef? None.

Sign Chart for the first derivative:

$f'(x)$: $-$ on $(-\infty,-1)$, 0 at -1, $+$ on $(-1,1)$, 0 at 1, $-$ on $(1,\infty)$

f: ↘ -1 ↗ 1 ↘

local min at $x=-1$; $(-1, f(-1)) : (-1,-1)$

local max at $x=1$; $(1, f(1)) : (1,1)$

f is increasing on: $(-1,1)$

f is decreasing on: $(-\infty,-1)$ & $(1,\infty)$

Study the second derivative:

$$f''(x) = \frac{4x^3 - 12x}{(x^2+1)^3} = \frac{4x(x^2-3)}{(x^2+1)^3} = \frac{4x(x-\sqrt{3})(x+\sqrt{3})}{(x^2+1)^3}$$

Sign Chart for the second derivative:

$f''(x)$: $-$ o $+$ o $-$ o $+$

f: $\cap$ $-\sqrt{3}$ $\cup$ 0 $\cap$ $\sqrt{3}$ $\cup$

$f'' = 0$ if $x = 0$, $x = \pm\sqrt{3}$

f'': undef never

Intervals of Increase:

Intervals of Decrease:

Concave up on:

Concave down on:

f is concave up on:

$(-\sqrt{3}, 0)$ & $(\sqrt{3}, \infty)$

f is concave down on:

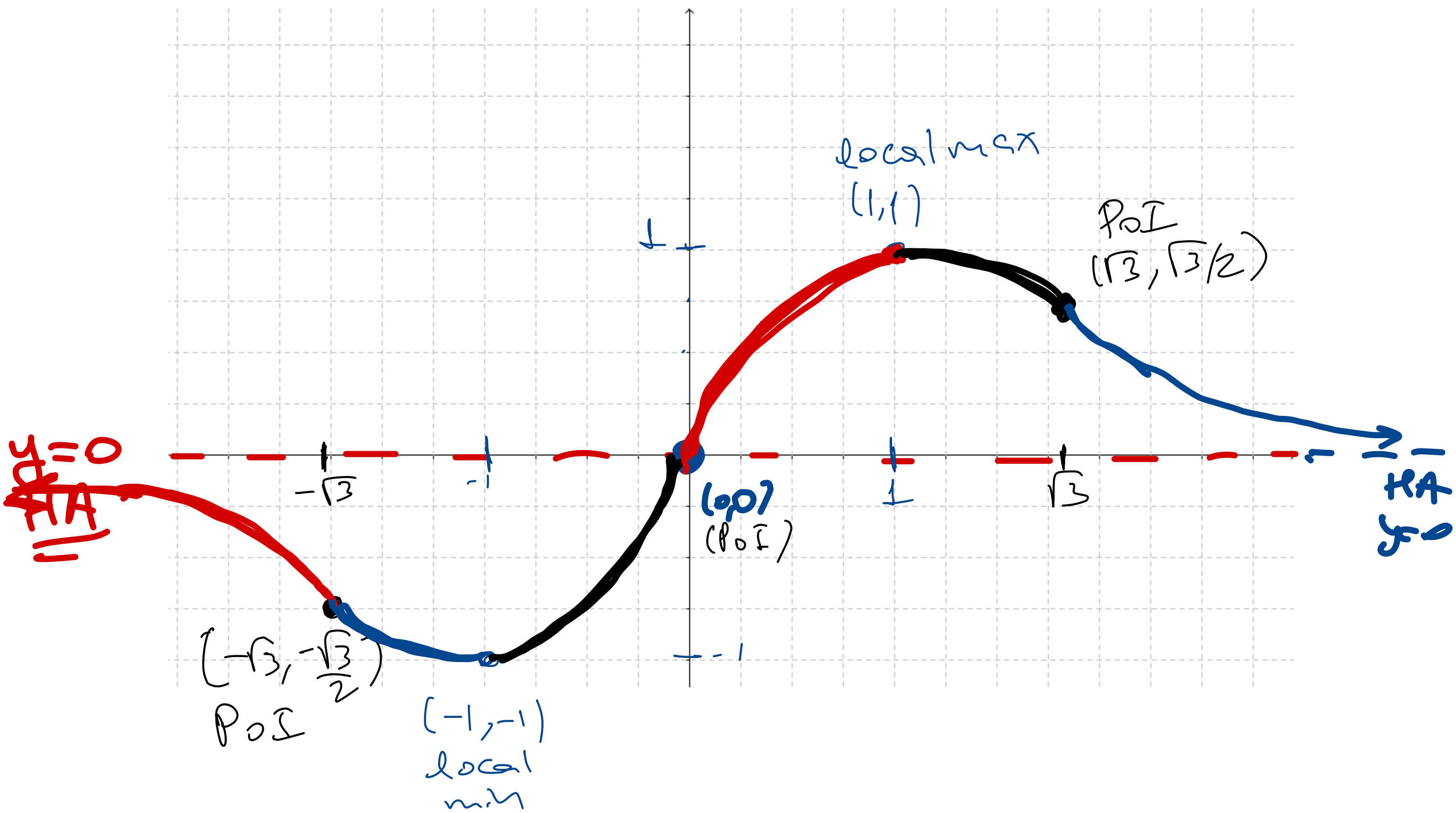
$(-\infty, -\sqrt{3})$ & $(0, \sqrt{3})$

PoI at: $x = 0$, $x = \sqrt{3}$, $x = -\sqrt{3}$

$(0, 0)$ $(\sqrt{3}, \sqrt{3}/2)$ $(-\sqrt{3}, -\sqrt{3}/2)$

$f(\sqrt{3}) = 2\sqrt{3}/4 = \sqrt{3}/2$

$\sqrt{3} \approx 1.7$



$$\frac{2x^2}{x^2 \ldots}$$

VA: 1 $\quad$ $x=1$ $\quad$ HA: $\boxed{y=1}$

Example: Graph $f(x)=\dfrac{2x^2}{(x-1)^2}$. Find the intervals of increase/decrease, concave up/down. Find critical points and classify them. Find and label any points of inflection.

$$f(-\tfrac{1}{2}) = \frac{2\cdot 1/4}{9/4} = \frac{1/2}{9/4} = \frac{1}{2}\cdot\frac{4}{9} = \frac{2}{9}$$

Solution:

Domain: $x \neq 1$ $\quad ((-\infty,1)\cup(1,\infty))$

Asymptotes: VA: $\boxed{x=1}$; HA: $\boxed{y=2}$

calc(..)

Intercepts: $(0,0)$

Critical Points (classify them):

$\frac{-}{+}$

$f'(x)=\dfrac{-4x}{(x-1)^3}$ $\quad \frac{+}{-} \quad \frac{=}{=}$

$f'=0 \Rightarrow \boxed{x=0}$ cp

f': undef $\Rightarrow \boxed{x=1}$ ← Not a c.p.

Sign Chart for the first derivative:

$f'(x)$	$- - - -$	0	$+++$	$\vert$	$- - - -$
f	↘	$\boxed{0}$	↗	1	↘

f increases on: $(0,1)$

f decreases on: $(-\infty,0)$ & $(1,\infty)$

local min at $x=0$; $(0,f(0))=\boxed{(0,0)}$.

concavity

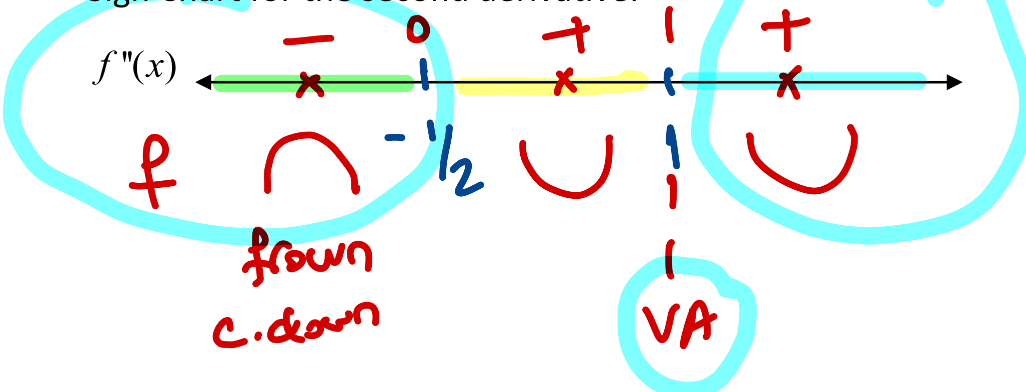
$f'' = 0$ if $x = \frac{-1}{2}$

f'': undef if $x = 1$

Study the second derivative:

$$f''(x) = \frac{8x+4}{(x-1)^4} = \frac{4(2x+1)}{(x-1)^4} \leftarrow +$$

Sign Chart for the second derivative:



Intervals of Increase:

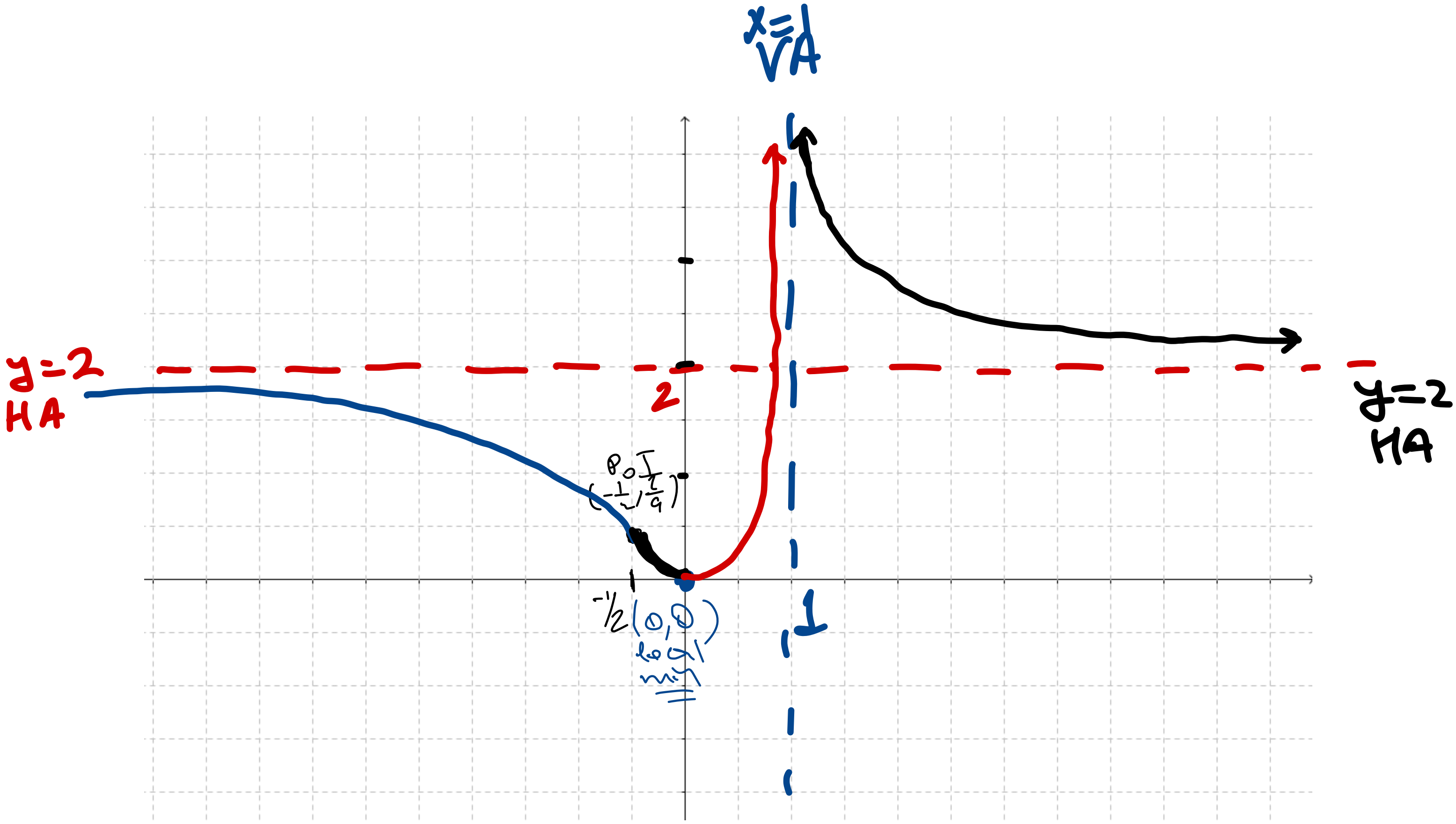
Intervals of Decrease:

Concave up on: $(-1/2, 1)$ & $(1, \infty)$

Concave down on: $(-\infty, -1/2)$

PoI at $x = -1/2$; $\left(-\frac{1}{2}, f\left(-\frac{1}{2}\right)\right)$

$\left(-\frac{1}{2}, \frac{2}{9}\right)$



$$\frac{2x^2}{(x-1)^2} = 2$$

EXERCISE: Graph the following functions!

Exercise: Graph $f(x)=\dfrac{2x^2}{x^2-1}$

next page

$$f'(x)=\frac{-4x}{\left(x^2-1\right)^2}$$

$$f''(x)=\frac{12x^2+4}{\left(x^2-1\right)^3}$$

Exercise: Graph $f(x)=\dfrac{2x^2-4x}{x^2-4}$

$$f'(x)=\frac{4}{(x+2)^2}$$

$$f''(x)=\frac{-8}{(x+2)^3}$$

Exercise: Graph $f(x)=\dfrac{x^2}{x-2}$

Asymptotes: (Note: This function has a slant asymptote!)

$$f'(x)=\frac{x^2-4x}{(x-2)^2}$$

$$f''(x)=\frac{8}{(x-2)^3}$$

Exercise:

$\rightarrow f(x) = \dfrac{2x^2}{x^2-1} = \dfrac{2x^2}{(x-1)(x+1)}$

$$f'(x) = \frac{-4x}{(x^2-1)^2}$$

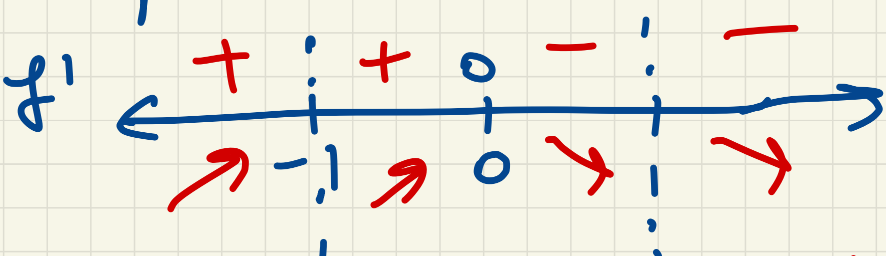
$$f''(x) = \frac{12x^2+4}{(x^2-1)^3} = \frac{4(3x^2+1)}{(x-1)^3(x+1)^3}$$

a) Asymptotes:

V.A: $x=1$, $x=-1$ HA: $y=2$

b) Critical points (classify them)

cp: $x=0$



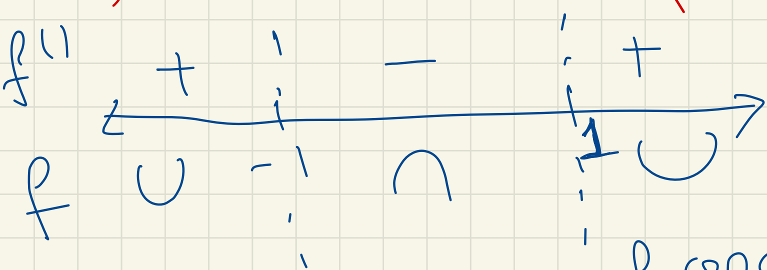
local max at $x=0$

c) increasing / decreasing

f is increasing on: $(-\infty,-1)$ & $(-1,0)$

decreasing on: $(0,1)$ & $(1,\infty)$

d) concave up/ down



f concaves up on: $(-\infty,-1)$ & $(1,\infty)$

e) PoI ?

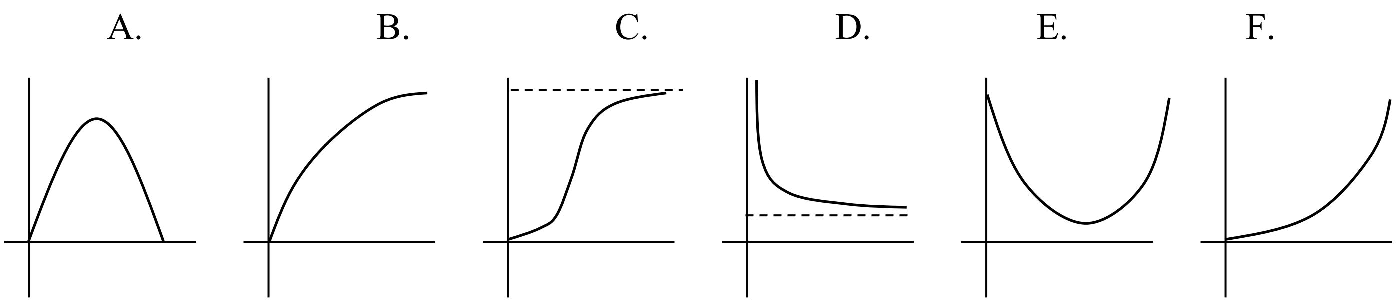
None

QUESTION:

What can you say about a function with these properties:

1. The domain is all real numbers except 3 and -3.
2. The function has vertical asymptotes at $x = 3$ and $x = -3$
3. The function is symmetric about the y-axis
4. $\lim_{x \to \infty} f(x) = -1$
5. $f(0) = 0, f(2) = 0, f(4) = 0$
6. $f'(x) < 0$ for $0 < x < 1$ and $x > 3$
7. $f'(x) > 0$ for $1 < x < 3$
8. $f''(x) < 0$ for $0 < x < 1/2$

Exercise: For the given functions, determine:



a) Which functions have a positive first derivative for all x?

b) Which functions have a negative first derivative for all x?

c) Which functions have a positive second derivative for all x?

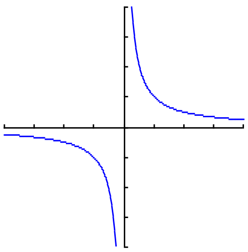
d) Which functions have a negative second derivative for all x?

POPPER#

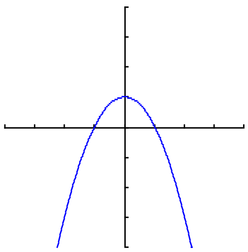
Match the function with its **first** derivative.

Functions:

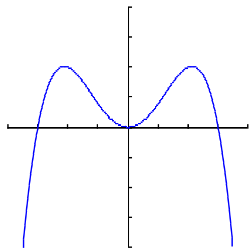
Q#



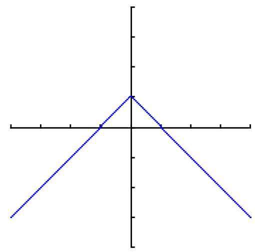
Q#



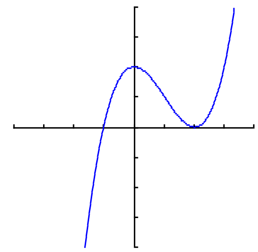
Q#



Q#

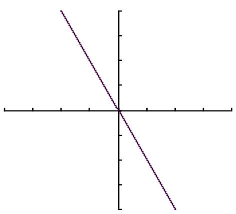


Q#

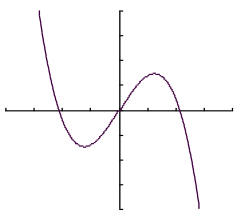


Derivatives:

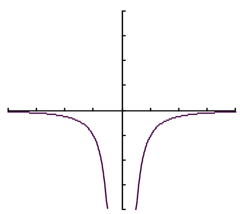
A.



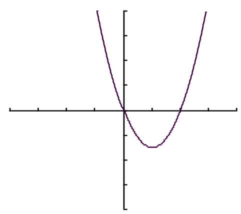
B.



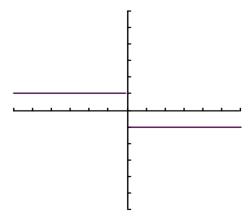
C.



D.

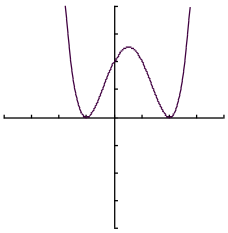


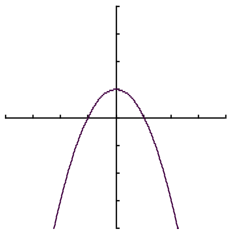
E.

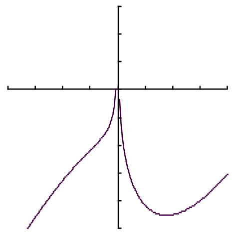


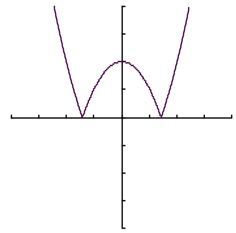
Match the function with its **second** derivative.

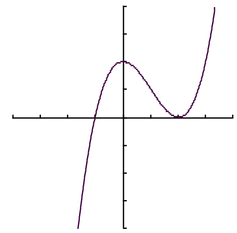
Functions:

Q# 

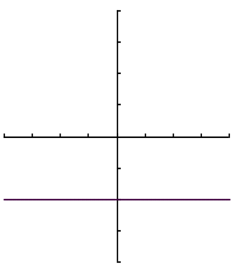
Q# 

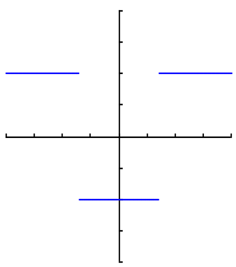
Q# 

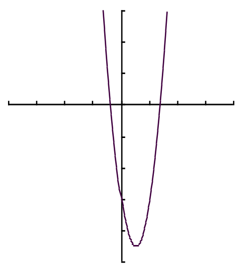
Q# 

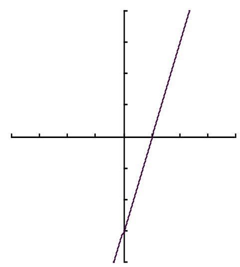
Q# 

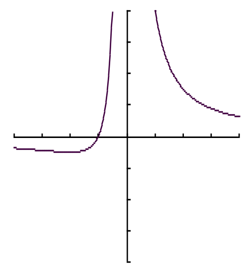
Second Derivatives:

A. 

B. 

C. 

D. 

E. 

POPPER #

Question# Which function has a vertical cusp at x = 1?

a) $f(x) = x^{1/3}$

b) $f(x) = (x-1)^{3/5}$

c) $f(x) = (x-1)^{2/5}$

d) $f(x) = (x+1)^{3/5}$

e) None

Question# Which function has a vertical tangent at x = 3?

a) $f(x) = 3 - x^{1/3}$

b) $f(x) = (x-3)^{3/5}$

c) $f(x) = (x-3)^{2/5}$

d) $f(x) = (x+3)^{4/5}$

e) None