Math 2413- Calculus I

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- Check CASA calendar for due dates.
- Bring "blank notes" to class. Completed notes will be posted after class.
- Do your best to attend every lecture and lab.
- Study after every lecture; work on the quiz covering the topic we cover on the lecture immediately afterwards. Retake your quizzes for more practice.
- Get help when you need help; bring your questions to the labs, or my office hours. We also have tutoring options on campus.
- Respect your friends in class; stay away from distractive behavior. Do your best to concentrate on the lecture.
- If you email me, mention the course code in the subject line. Email is the best way to communicate with me outside of class. Teams chat messages are not monitored or replied to.

Question# **Which of the following functions is one-to-one?**

A) $f(x) = x^2$ **B)** $f(x) = \sin(x)$ **C)** $f(x) = x^3$ **D)** $f(x) = |x|$ **E) None**

Chapter 4 – The Transcendental Functions

Section 4.1 – Inverse Functions

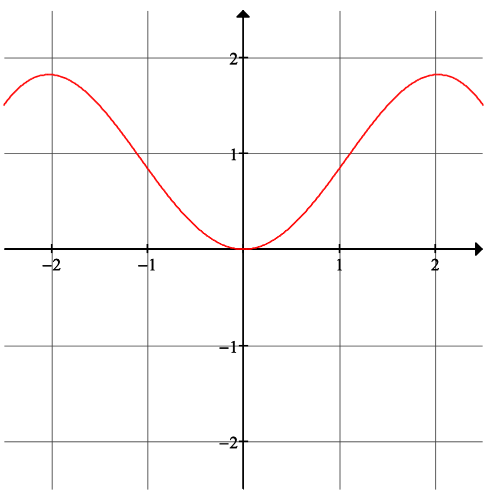
RECALL (ALGEBRA)—

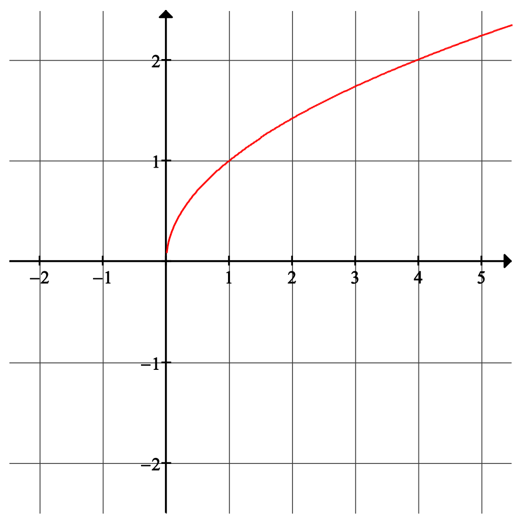
Definition: A function is said to be **one-to-one** if there are no two distinct numbers in the domain of f that produce the same value.

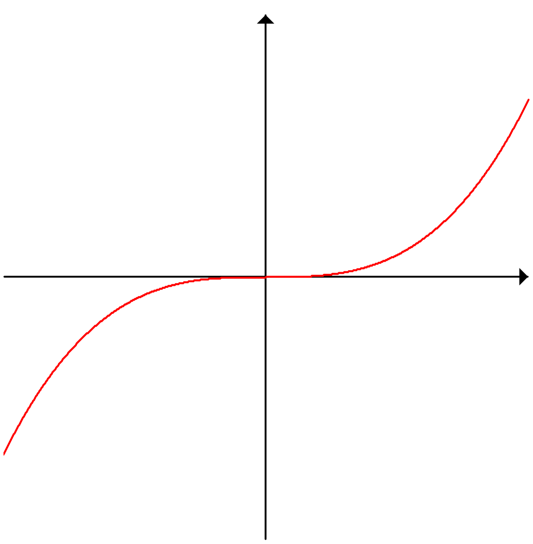
$$f(x_1) = f(x_2) \text{ implies } x_1 = x_2.$$

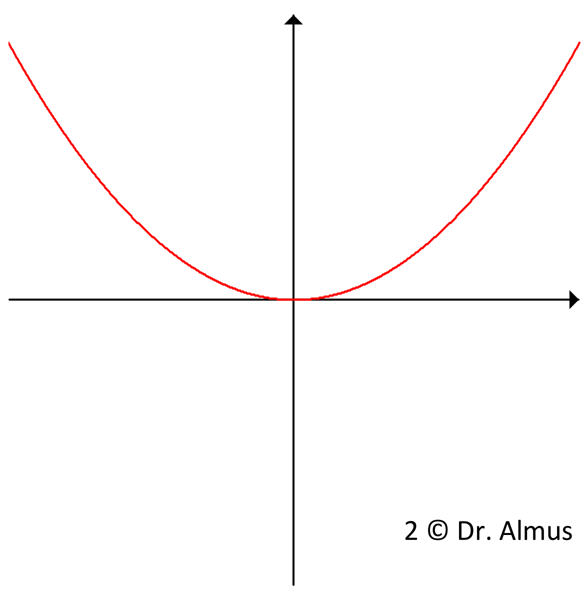
In other words, two different x values cannot have the same y value.

Which of the following functions are one-to-one? (Use the horizontal line test!)









Using the definition to prove whether a function is 1-1 or not:

Let f be a function. It is a 1-1 function if: $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Example: Show that $f(x) = (x + x^2)^3$ is NOT one-to-one.

Example: Show that $f(x) = \sqrt[3]{5x-2}$ is one-to-one.

Solution: Assume $f(x_1) = f(x_2)$;

$$\sqrt[3]{5x_1 - 2} = \sqrt[3]{5x_2 - 2} \xrightarrow{\text{take cubes of both sides}} 5x_1 - 2 = 5x_2 - 2 \xrightarrow{\text{add 2 to both sides}} 5x_1 = 5x_2 \xrightarrow{\text{divide both sides by 5}} x_1 = x_2.$$

Hence, the function is 1-1.

Definition:

Let f be a one-to-one function. There exists a unique function f^{-1}, called the **inverse of** f, such that for every x in the domain of f, we have:

$$\left(f^{-1} \circ f\right)(x) = x.$$

We will use the notation $f^{-1}(x)$ to denote the inverse of $f(x)$.

Remark: $f^{-1}(x) \neq \dfrac{1}{f(x)}$

For example: $f(x) = x^3 \rightarrow f^{-1}(x) = x^{1/3}$; as you see, $f^{-1}(x) \neq \dfrac{1}{f(x)}$.

Note: Domain of f becomes the range of f^{-1}

Range of f becomes the domain of f^{-1}

FACT: $f(a) = b \rightarrow f^{-1}(b) = a$

If (a,b) is a point on the graph of $f(x) \rightarrow (b,a)$ is a point on the graph of $f^{-1}(x)$.

Example: If f is an invertible function with $f(0) = 2,$ and $f(2) = 5$; find $f^{-1}(2) = ?$

How do we find the formula for the inverse of a function?

1. Start with $y = f(x)$

2. **Switch** the x's and y's.

3. Solve for y; get something like: $y = g(x)$

4. The function g is the inverse of *f*. $g(x) = f^{-1}(x)$

We can only do this for simple functions.

Exercise: Is $f(x) = 2x - 5$ invertible? If so, find its inverse.

Exercise: Find the inverse of $f(x) = \dfrac{x+2}{x-1}$.

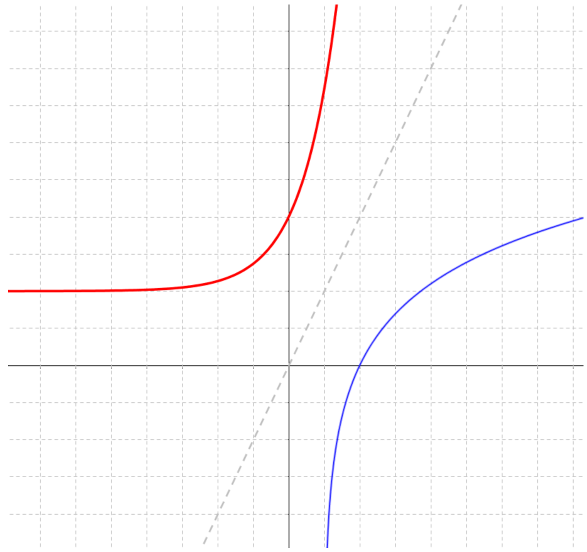
Exercise: An example of real life inverse functions:

The formula $f(x) = \dfrac{9}{5}x + 32$ is used to convert from degrees Celsius to degrees Fahrenheit.

The formula $g(x) = \dfrac{5}{9}(x - 32)$ is used to convert from degrees Fahrenheit to degrees Celsius. Verify that these functions are inverses of each other.

How are functions related to their inverses?

Algebraic:



Geometric:

NEW (CALCULUS): A test for being invertible --

Definition: A function is **monotonic** if it is <u>always</u> increasing or <u>always</u> decreasing on its domain.

Theorem:

If f is monotonic, then f is an invertible function.

Recall:

If $f' > 0$, then f is increasing.

If $f' < 0$, then f is decreasing.

That is:

- If $f' > 0$ on the domain of f, then f is **monotonic (and hence, invertible).**
- If $f' < 0$ on the domain of f, then f is **monotonic (and hence, invertible).**

Example: Show that $f(x) = x^3 + 5x$ is invertible.

Example: Show that $f(x)=\dfrac{x+1}{x-1}$ is invertible.

Example: Let $f\left(x\right)=\dfrac{1}{3}x^3-x^2+kx$. For what values of k is f(x) invertible?

Finding the Derivative of the Inverse Function

Theorem:

If $f(x)$ is **continuous and invertible** then $f^{-1}(x)$ is continuous.

Theorem:

If $f(x)$ is **differentiable and invertible**, and $f'(x)$ is **nonzero**, then $f^{-1}(\mathrm{x})$ is differentiable.

Also, we have a shortcut to find the derivative at a certain point:

$f(a)=b$ and $f'(a)\neq 0$, then we can find the derivative of f^{-1} at $x=b$ using:

$$\left(f^{-1}\right)'(b)=\frac{1}{f'(a)}.$$

Example: $f(x) = x^3$; we know $f(2) = 8$.

If you want to find $\left[f^{-1}\right]'(8)$, we can use the shortcut:

$$\left[f^{-1}\right]'(8) = \frac{1}{f'(2)} = \frac{1}{3(2)^2} = \frac{1}{12}.$$

CHECK this using the longer way (first find the inverse, and then take its derivative):

$$f(x) = x^3 \to f^{-1}(x) = x^{1/3}$$

Now take the derivative: $\left[f^{-1}(x)\right]' = \left[x^{1/3}\right]' = \frac{1}{3}x^{-2/3}$

$$\left[f^{-1}\right]'(8) = \frac{1}{3}(8)^{-2/3} = \frac{1}{12}$$

Example: We showed earlier that $f(x) = x^3 + 5x$ was invertible.

Find $\left(f^{-1}\right)'(6)$.

Example: Given $f(x) = 4x + \cos(x) - \sin(x)$ for $x \in \left[0, \frac{\pi}{2}\right]$,

find $\left[f^{-1}\right]'(\pi) = ?$

(Hint: $f\left(\frac{\pi}{4}\right) = \pi$)

Example: If f is invertible, and $f(1) = 2$, $f(3) = 1$, $f'(1) = 4$, $f'(3) = 5$, $f'(2) = 6$,

find $\left(f^{-1}\right)'(1)$.

Exercise: Let $f(x) = x^5 + 2x^3 + 2x$. Give an equation for the tangent line to the graph of $f^{-1}(\text{x})$ at the point (–5, –1).

Exercise: Given: $f(x) = x^5 - 4x^3 + 6x$ is invertible.

Find $\left(f^{-1}\right)'(12) = ?$.

POPPER#

Question# Given: the slope of the tangent line to *f* at (5,4) is 7, and at (1,5) is 8. Find the slope of the line tangent to the curve f^{-1} at x = 5.

A) 1/8
B) 1/7
C) 7
D) 8
E) None

Question# Given $\mathrm{f}\left(x\right)=x^3+5x$, find $\left[f^{-1}\right]'(18)$ if possible.

A) 1/18 B) 1/17 C) -1/17 D) 1/2 E) None