Math 2413- Calculus I

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- Check CASA calendar for due dates.
- Bring "blank notes" to class. Completed notes will be posted after class.
- Do your best to attend every lecture and lab.
- Study after every lecture; work on the quiz covering the topic we cover on the lecture immediately afterwards. Retake your quizzes for more practice.
- Get help when you need help; bring your questions to the labs, or my office hours. We also have tutoring options on campus.
- Respect your friends in class; stay away from distractive behavior. Do your best to concentrate on the lecture.
- If you email me, mention the course code in the subject line. Email is the best way to communicate with me outside of class. Teams chat messages are not monitored or replied to.

Section 4.3 – Logarithmic Function

Definition: The logarithmic function with base a $(a > 0, a \neq 1)$ is the function $\log_a(x)$ that satisfies:

$$\log_a\left(a^x\right) = x \quad \text{for all } x \in (-\infty, \infty),$$

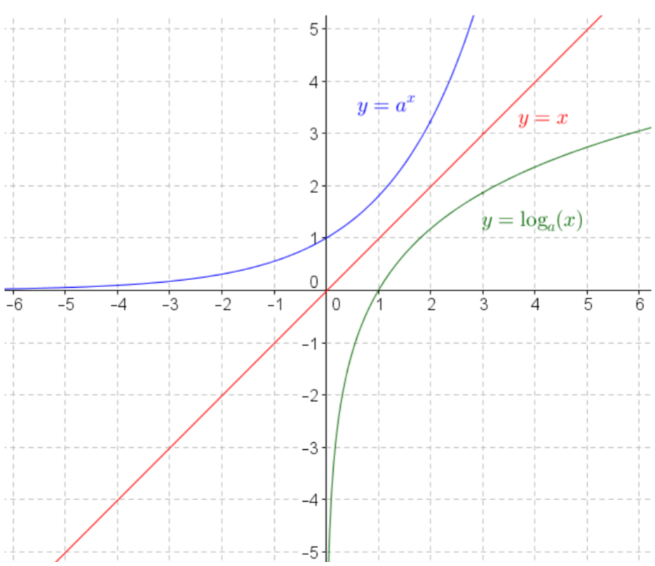
$$a^{\log_a(x)} = x, \text{ for all } x \in (0, \infty).$$

Base 10: Common logarithm

Base e: Natural logarithm; $y = \log_e(x) = \ln(x)$.

The function $y = \log_a(x)$ is the **inverse** of the exponential function $y = a^x$.

Domain: $(0, \infty)$, Range: $(-\infty, \infty)$



RECALL -- Converting expressions:

$a^b = c \to \log_a(c) = b$

$2^5 = 32 \to \log_2(32) = 5$

$1000 = 10^3 \to \log_{10}(1000) = 3$

$\log_a(c) = b \to c = a^b$

$\log_4(16) = 2 \to 16 = 4^2$

$\log_3(x) = 5 \to x = 3^5$

$\log_2(x+4) = 10 \to x + 4 = 2^{10}$

$\ln(x) = 2 \to x = e^2$

Laws of Logarithms

1. $\log_a(x \cdot y) = \log_a x + \log_a y$
2. $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$
3. $\log_a(x^p) = p \cdot \log_a x$

very important

$\log_2(8) = 3$

$\log(1000) = 3$

$\log_2(32) = 5$

$2^? = 32$

$\ln(e) = 1$

$\ln(e^2) = 2$

$\ln(1) = 0$

very important!

Example: Expand using the Laws of Logarithms:

$$\ln\left(\frac{x^3\sqrt{x+2}}{(x+1)^2}\right) = \ln(x^3) + \ln(\sqrt{x+2}) - \ln(x+1)^2$$

$$= 3\ln(x) + \frac{1}{2}\ln(x+2) - 2\ln(x+1)$$

Example: Simplify using the Laws of Logarithms:

$2\log(x) - \log(x+1) + 4\log(x+2)$

$$= \log(x^2) - \boxed{\log(x+1)} + \log(x+2)^4$$

$$= \log\left(\frac{x^2 \cdot (x+2)^4}{(x+1)}\right)$$

$$\log(A) - \log(B) + \log(C) = \log\left(\frac{A \cdot C}{B}\right)$$

Differentiating Natural Logarithmic Function:

Fact: $$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

If u is a function of x, then using the chain rule, we get:

$$\frac{d}{dx}(\ln(u)) = \frac{1}{u} \cdot \frac{du}{dx} = \frac{u'}{u}$$

$\ln(u) \xrightarrow{\text{derivative}} \frac{u'}{u}$

Example: Differentiate the following functions.

$y = 5\ln(x)$

$\Rightarrow y' = 5 \cdot \frac{1}{x} = \frac{5}{x}$

$y = \ln(4x)$

$\Rightarrow y' = \frac{4}{4x} = \frac{1}{x}$

$y = \ln(x^6 + 1)$

$\Rightarrow y' = \frac{u'}{u'} = \frac{6x^5}{x^6+1}$

$y = \ln(x^3 + x)$

$\Rightarrow y' = \frac{3x^2+1}{x^3+x}$

$y=\ln(\sin(x)) \Rightarrow y' = \frac{\cos(x)}{\sin(x)} = \cot(x)$

$f(x)=\ln\left(\sqrt{x^2+4}\right)$ Better: $f(x) = \frac{1}{2}\ln(x^2+4)$

$f'(x) = \frac{1}{2}\cdot\frac{2x}{x^2+4} = \frac{x}{x^2+4}$

Example: $f(x)=e^{2x}\ln(4x)$; $f'(2)=?$

product rule!

$f'(x) = (e^{2x}\cdot 2)\cdot\ln(4x) + e^{2x}\cdot\frac{4}{4x}$

$f'(2) = e^{4}\cdot 2\cdot\ln(8) + e^{4}\cdot\frac{1}{2}$

$f'(2) = e^{4}\left(2\ln(8) + \frac{1}{2}\right)$ //

Note: $2\ln(8) = \ln(8^2) = \ln(64)$

or $2\ln(8) = 2\ln(2^3) = 6\ln(2)$

$u^3 \xrightarrow{\text{deriv.}} 3 \cdot u^2 \cdot u'$ ← chain rule

Example: $f(x) = \ln^3(5x)$; $f'(x) = ?$

$f(x) = [\ln(5x)]^3$

deriv.

$\Rightarrow f'(x) = 3[\ln(5x)]^2 \cdot \left[\frac{5}{5x}\right]$

* Rewrite using properties of ln first!

Example: Find the derivative of $y = \ln\left(\frac{x^4(x+1)}{\sqrt{x^3+2}}\right)$

$y = \ln(x^4) + \ln(x+1) - \ln(\sqrt{x^3+2})$

$y = 4\ln(x) + \ln(x+1) - \frac{1}{2}\ln(x^3+2)$

↓ derivative

$y' = 4 \cdot \frac{1}{x} + \frac{1}{x+1} - \frac{1}{2} \cdot \frac{3x^2}{x^3+2}$

Different Bases

Change of base formula: $\boxed{\log_a x = \dfrac{\ln x}{\ln a}}$

Fact: $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x \ln a}$

If u is a function of x, then using the chain rule, we get:

$$\frac{d}{dx}(\log_a u) = \frac{1}{u \ln a} \cdot \frac{du}{dx} = \frac{u'}{u \ln a}$$

Example: Find the derivatives.

$y = \log_2(x)$ $\Rightarrow$ $y' = \dfrac{1}{x \ln(2)}$

$y = \log_7(5x+2)$

$\Rightarrow$ $y' = \dfrac{5}{(5x+2) \cdot \ln(7)}$

$y = \log_3(4x^3 + 2x)$ $\Rightarrow$ $y' = \dfrac{12x^2 + 2}{(4x^3+2x) \cdot \ln(3)}$

$y = \log(\cos(4x)) = \log_{10}(\cos(4x))$

Base 10

$\Rightarrow$ $\boxed{y' = \dfrac{-4\sin(4x)}{\cos(4x) \cdot \ln(10)}}$ ← final answer.

$$y = \underbrace{\frac{1}{\ln(10)}} \cdot \ln(\cos(4x))$$

$$y' = \frac{1}{\ln(10)} \cdot \frac{-4\sin(4x)}{\cos(4x)}$$

$$x^n \xrightarrow[\text{power rule}]{\text{deriv.}} n \cdot (x)^{n-1}$$

$$(5x+2)^3 \xrightarrow[\text{power rule chain rule}]{\text{deriv.}} 3(5x+2)^2 \cdot 5$$

$$3^x \xrightarrow[\text{exp. rule}]{\text{deriv.}} 3^x \cdot \ln(3)$$

$$3^{5x+2} \longrightarrow 3^{5x+2} \cdot \ln(3) \cdot 5$$

$(x+1)^{5x+2} \xrightarrow{\text{derivative}}$?

Base: x's

exponent: x's

very important!

METHOD: Logarithmic Differentiation

This is a method we choose to use while taking derivatives of functions that are otherwise difficult to differentiate. We introduce a logarithmic expression, and use properties of logarithm to make the problem easier to differentiate.

We start with: $y = f(x)$; we need $y' = f'(x)$ but we don't know how to do this directly.

Introduce logarithms: $y = f(x) \xrightarrow{\text{take ln's of both sides}} \ln(y) = \ln(f(x))$

Now, take derivatives of both sides:

$$\frac{y'}{y} = \frac{d}{dx}\left[\ln(f(x))\right]$$

$$y' = y \cdot \left(\frac{d}{dx}\left[\ln(f(x))\right]\right)$$

Examples where this method might be used:

$\left(x^2+1\right)^{4x+2}$ **---** *(expression with a power, but exponent is not a number, so power rule doesn't apply. This is the only method that helps at this point).*

$\dfrac{x^3\sin(2x)}{\sqrt{x^2+4}}$ **---** *(complicated expression that would need product rule, quotient rule, and chain rule – would be difficult to take the derivative directly, so we may CHOOSE to use this method.)*

$$A = B \implies \ln(A) = \ln(B)$$

Example: Find the derivative of $\boxed{y} = (4x+1)^{\sin(x)}$.

Question: $y' = ?$

Logarithmic differentiation.

Step 1: Rewrite: $\ln(y) = \ln\left(\underline{(4x+1)^{\sin(x)}}\right)$

Rewrite using properties of ln:

$$\ln(y) = \underbrace{\sin(x) \cdot \ln((4x+1))}$$

↓ Take derivatives of both sides (product rule on the right)

$$\frac{y'}{y} = \cos(x) \cdot \ln(4x+1) + \sin(x) \cdot \frac{4}{4x+1}$$

solve for y' (multiply both sides by y)

$$y' = \boxed{y \cdot}\left[\cos(x)\ln(4x+1) + \sin(x) \cdot \frac{4}{4x+1}\right]$$

↓ formula for y (given in the problem)

$$y' = (4x+1)^{\sin(x)} \cdot \underbrace{\left[\cos(x)\ln(4x+1) + \sin(x)\frac{4}{4x+1}\right]}_{\text{all } x\text{'s}.}$$

$$\ln(A^{B}) = B\ln(A)$$

$$\ln(x^5) = 5\ln(x)$$

$$\ln\left((5x+2)^{2x+1}\right)$$

$$= (2x+1)\cdot\ln(5x+2)$$

$(x\text{'s} \ldots)^{x\text{'s}\ldots}$ method: log. differentiation

Example: Find the slope of the tangent line to the curve $y = (5x+2)^{2x+1}$ at $x = 0$.

$$y = (5x+2)^{2x+1}$$

$\Rightarrow$ Take ln's of both sides:

$$\ln(y) = \ln(5x+2)^{2x+1}$$

property of ln: $\ln(A^B) = B\ln(A)$

$$\Rightarrow \ln(y) = (2x+1) \cdot \ln(5x+2)$$

Take derivatives of both sides

$$\Rightarrow \frac{y'}{y} = 2 \cdot \ln(5x+2) + (2x+1) \cdot \frac{5}{5x+2}$$

$$\Rightarrow y' = y \cdot \left[2\ln(5x+2) + \frac{5(2x+1)}{5x+2} \right]$$

formula

$$\Rightarrow y' = (5x+2)^{2x+1} \cdot \left[2\ln(5x+2) + \frac{5(2x+1)}{5x+2} \right]$$

$$m_{\substack{\text{tangent} \\ x=0}} = (2)^{0+1} \cdot \left[2 \cdot \ln(2) + \frac{5 \cdot 1}{2} \right]$$

slope $= 2 \cdot \left[2\ln(2) + \frac{5}{2}\right] = 4\ln(2) + 5$ //

Example: Find the derivative of $y = \dfrac{x\sin(2x)}{\sqrt{x^3+1}}$ using logarithmic differentiation.

must use it

$$y = \frac{x \cdot \sin(2x)}{\sqrt{x^3+1}}$$

$$\Rightarrow \ln(y) = \ln\left(\frac{x \cdot \sin(2x)}{\sqrt{x^3+1}}\right)$$

use properties of ln to rewrite

$$\Rightarrow \ln(y) = \ln(x) + \ln(\sin(2x)) - \ln\left(\sqrt{x^3+1}\right)$$

$$\ln(y) = \ln(x) + \ln(\sin(2x)) - \frac{1}{2}\ln(x^3+1)$$

take derivatives of both sides:

$$\frac{y'}{y} = \frac{1}{x} + \frac{2\cos(2x)}{\sin(2x)} - \frac{1}{2} \cdot \frac{3x^2}{x^3+1}$$

$$\Rightarrow y' = \frac{x \cdot \sin(2x)}{\sqrt{x^3+1}} \cdot \left[\frac{1}{x} + 2\cot(2x) - \frac{3x^2}{2(x^3+1)}\right]$$