Math 2413- Calculus I

Dr. Melahat Almus

Email: malmus@uh.edu

- Check CASA calendar for due dates.
- Bring "blank notes" to class. Completed notes will be posted after class.
- Do your best to attend every lecture and lab.
- Study after every lecture; work on the quiz covering the topic we cover on the lecture immediately afterwards. Retake your quizzes for more practice.
- Get help when you need help; bring your questions to the labs, or my office hours. We also have tutoring options on campus.
- Respect your friends in class; stay away from distractive behavior. Do your best to concentrate on the lecture.
- If you email me, mention the course code in the subject line. Email is the best way to communicate with me outside of class. Teams chat messages are not monitored or replied to.

Guillaume de l'Hôpital

Born	1661 Paris, France
Died	2 February 1704 Paris, France

1696

Section 5.3 – L'Hospital's Rule

In this section, we will re-visit the topic of limits. We covered many limit problems in Chapter 2. Here, we will see a theorem that helps us answering limit questions where direct substitution produced an "indeterminate form":

$$\lim_{x \to 2}\left(\frac{x-2}{x^2-4}\right),$$

$$\lim_{x \to 3}\left(\frac{\sqrt{x}-3}{x-9}\right),$$

$$\lim_{x \to 0}\left(\frac{1}{x}-\frac{1}{x^2}\right) = ?, \text{ etc.}$$

If the expression obtained after this substitution does not provide sufficient information to determine the original limit, then the expression is called an **indeterminate form**. More specifically, an indeterminate form is a mathematical expression involving at most two of : 0, 1 or ,∞ obtained by applying the algebraic limit theorem in the process of attempting to determine a limit, which fails to restrict that limit to one specific value or infinity, and thus does not determine the limit being sought.

There are 7 indeterminate forms commonly found in the literature:

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 1^{\infty}, \quad \infty^0, \quad 0^0, \quad 0 \bullet \infty, \quad \infty - \infty$$

If direct substitution produces one of the forms, one needs to work harder on that problem to figure out the answer. Sometimes, that extra work is "algebra" (as we did in Chapter 2); simplifying before substitution might solve the issue of having these forms. We will see other methods in this section.

Indeterminate Forms – 0/0 and ∞/∞

L'Hospital's Rule (0/0)

Let f, g be two functions differentiable on an open interval I that contains c and suppose that $g'(x) \neq 0$ on I (except possibly at c).

If $\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = 0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)},$$

provided that the limit on the right hand side exists.

On the other hand, if $\frac{f'(x)}{g'(x)} \to \pm\infty$ as $x \to c$, then $\frac{f(x)}{g(x)} \to \pm\infty$ as well.

When you answer limit questions:

First, use direct substitution, if you get an indeterminate form **0/0 or ∞/∞, state that you will use L'Hospital's rule.** This is important; you must mention the name of the theorem in your work.

Revisit: $\lim_{x\to 2}\left(\dfrac{x-2}{x^2-4}\right) = ?$

Chapter 2 (using algebra)

$$\lim_{x\to 2}\left(\frac{x-2}{x^2-4}\right)\xrightarrow{\text{0/0; use algebra}}\lim_{x\to 2}\left(\frac{x-2}{x^2-4}\right)=\lim_{x\to 2}\left(\frac{x-2}{(x-2)(x+2)}\right)=\lim_{x\to 2}\left(\frac{1}{x+2}\right)\overset{plug\ \text{in}}{=}\frac{1}{4}$$

Chapter 5 (using derivatives)

$$\lim_{x\to 2}\left(\frac{x-2}{x^2-4}\right)\xrightarrow{\text{0/0; use L'Hospital's rule}}\lim_{x\to 2}\left(\frac{x-2}{x^2-4}\right)\overset{L'H.rule}{=}\lim_{x\to 2}\left(\frac{1}{2x}\right)\overset{plug\ \text{in}}{=}\frac{1}{4}$$

This method saves a lot of time; especially when algebra requires a lot of factoring.

Example 1: Evaluate the limit if it exists.

$$\lim_{x\to 1}\left(\frac{x^2-3x+2}{x^3-1}\right)=$$

$\dfrac{1-3+2}{1-1}=\dfrac{0}{0}$

L'Hospital's rule: $=\lim_{x\to 1}\left(\dfrac{2x-3}{3x^2}\right)=\dfrac{2\cdot 1-3}{3\cdot 1^2}=-\dfrac{1}{3}$

Example 2: Evaluate the limit if it exists.

$$\lim_{x\to 0}\left(\frac{e^x - x - 1}{5x^2}\right) = \frac{e^0 - 0 - 1}{5\cdot 0^2} = \frac{0}{0}$$ can use L'H.'rule

plug in 0

L'H's rule $= \lim_{x\to 0}\left(\frac{e^x - 1}{10x}\right) = \frac{e^0 - 1}{10\cdot 0} = \frac{0}{0}$

plug in 0

L'H's rule $= \lim_{x\to 0}\left(\frac{e^x}{10}\right) \overset{?}{=} \frac{e^0}{10} = \boxed{\frac{1}{10}}$

Example 3: Evaluate the limit if it exists.

$$\lim_{x\to \pi}\frac{1+\cos(x)}{x-\pi} = \frac{1+\cos(\pi)}{\pi - \pi} = \frac{1+-1}{0} = \frac{0}{0}$$

L'H $= \lim_{x\to\pi}\frac{-\sin(x)}{1} = \frac{-\sin(\pi)}{1} = \frac{0}{1}$

plug in

$= \boxed{0}$

$(x-\pi)' = 1$

warning DO NOT MISUSE IT

ex:

$$\lim_{x\to 2} \frac{x+1}{5x-1} \overset{L'H}{=} \lim_{x\to 2} \left(\frac{1}{5}\right) = \frac{1}{5}$$

↓ plug in

$$\frac{2+1}{5\cdot 2-1} = \frac{3}{9} = \boxed{\frac{1}{3}}$$

Revisit Section ~~2.6~~ 1.6:

We learned a fact in Chapter 2: $\lim_{x\to 0}\left(\frac{\sin(ax)}{bx}\right)=\frac{a}{b}$.

Now, we can use L'Hospitals rule instead:

$$\lim_{x\to 0}\left(\frac{\sin(ax)}{bx}\right)\overset{L'H.rule}{=}\lim_{x\to 0}\left(\frac{a\cos(ax)}{b}\right)\overset{plugin}{=}\frac{a}{b}$$

L'Hospital's Rule (∞/∞)

Let f, g be two functions differentiable on an open interval I that contains c and suppose that $g'(x)\neq 0$ on I (except possibly at c).

If $f(x)\to\pm\infty$ and $g(x)\to\pm\infty$, then

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)},$$

provided that the limit on the right hand side exists.

On the other hand, if $\frac{f'(x)}{g'(x)}\to\pm\infty$ as $x\to c$, then $\frac{f(x)}{g(x)}\to\pm\infty$ as well.

Example 4: Evaluate the limit if it exists.

$$\lim_{x\to\infty}\left(\frac{x^2+x}{3x^2+4}\right) = \frac{\infty}{\infty}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{2x+1}{6x} = \frac{\infty}{\infty}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{2}{6} = \boxed{\frac{1}{3}}$$

$[\ln(x)]^2 \qquad u^2$

Example 5: Evaluate the limit if it exists.

$$\lim_{x\to\infty}\left(\frac{\ln^2(x)}{x}\right) = \frac{\infty}{\infty}$$

as $x\to\infty$, $\ln(x)\to\infty$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{2\cdot\ln(x)\cdot\frac{1}{x}}{1}$$

$$\overset{\text{rewrite}}{=} \lim_{x\to\infty} \frac{2\ln(x)}{x} = \frac{\infty}{\infty}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{2\cdot\frac{1}{x}}{1} \underset{\text{rewrite}}{=} \lim_{x\to\infty} \frac{2}{x} = \boxed{0}$$

EXERCISE: Evaluate the limit if it exists.

$$\lim_{x\to\infty} \frac{1+\sqrt{x}}{e^x} =$$

$$\lim_{x\to\infty} \frac{x+x^2}{\ln x} =$$

Indeterminate Forms

These are all indeterminate:

$\frac{0}{0}$ $\frac{\infty}{\infty}$ 1^{∞} ∞^{0} 0^{0} $0 \bullet \infty$ $\infty - \infty$

These are "determinate".

$\infty + \infty \to \infty$

$-\infty - \infty \to -\infty$

$0^{\infty} \to 0$

$0^{-\infty} \to \infty$

Note: $\infty - \infty$ is not zero; infinity is not a real number; you cannot use cancellation properties. However, if you have $k \bullet \infty$ where k is a nonzero real number, then that expression is not indeterminate: $2 \cdot \infty = \infty$, or $5 \cdot \infty = \infty$, etc.

By definition, $1^{0} = 1$; but 0^{0} is an indeterminate form.

CASE 1: Indeterminate Forms $0 \cdot \infty$, $\infty - \infty$:

Rewrite the function so that you get 0/0 or ∞ / ∞. And then, use L'Hospital's Rule.

$e^{-\infty} \to 0$

$0 \bullet \infty$

Example 6: $\lim_{x \to \infty} e^{-x}\sqrt{x} = 0 \cdot \infty$ indet. form

$=$

Rewrite: $\lim_{x \to \infty} \frac{\sqrt{x}}{e^x} = \frac{\infty}{\infty}$ use L'H. rule

$\overset{L'H.}{=} \lim_{x \to \infty} \frac{\frac{1}{2\sqrt{x}}}{e^x}$

$\overset{\text{rewrite}}{=} \lim_{x \to \infty} \frac{1}{2\sqrt{x} \cdot e^x} = \boxed{0}$

Fact: $\frac{1}{\infty} \to 0$

important ★

Example7: $\lim_{x\to\infty}\left(x\sin\left(\frac{4}{x}\right)\right)=$ $\infty\cdot\sin(0)=\infty\cdot 0$

rewrite: $\lim_{x\to\infty}\frac{\sin\left(\frac{4}{x}\right)}{\frac{1}{x}}=\frac{0}{0}$

L'H rule $=\lim_{x\to\infty}\frac{\cos\left(\frac{4}{x}\right)\cdot\left(-\frac{4}{x^2}\right)}{-\frac{1}{x^2}}=\lim_{x\to\infty}\left(-4\cos\left(\frac{4}{x}\right)\right)$ rewrite

$=-4\cdot\cos(0)=\boxed{-4}$

$\infty-\infty$

Example 8: $\lim_{x\to 2^+}\left(\frac{8}{x^2-4}-\frac{x}{x-2}\right)=$

$$AB = \frac{B}{\frac{1}{A}}$$

product

ratio

$$x \cdot B = \frac{B}{\frac{1}{x}}$$

Example 9: $\lim_{x\to 0^+}\left(\frac{1}{x}-\csc(x)\right)=$

Exercise: $\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=$

CASE 2: Indeterminate Powers 0^0, 1^∞ and ∞^0 -- Use a logarithmic "trick".

1^∞

Example 10: $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x =$

Exercise: Make sure you can solve these! This will be important in Calculus 2.

$$\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{3x}=$$

$$\lim_{x\to\infty}\left(1-\frac{4}{x}\right)^{2x}=$$

Generalize this idea: $\lim_{x\to\infty}\left(1-\frac{a}{x}\right)^{bx}=?$

Example 11: $\lim_{x\to 0^+}(1+x)^{2/x}=$

∞^0

Example 12: $\lim_{x\to\infty}\left(3^x+4^x\right)^{\frac{1}{x}}=$

Exercise: $\lim_{x\to\infty}\left(x^3+1\right)^{\frac{1}{\ln(x)}}=$

0^0

Example 13: $\lim_{x\to 0^+} \left(\sin(x)\right)^x =$

In summary:

Indeterminate forms: 0/0 or ∞/∞ :

$\rightarrow$ Use L'Hospital's rule.

Indeterminate Forms $0\cdot\infty$, $\infty-\infty$:

$\rightarrow$ Rewrite the function so that you get 0/0 or ∞/∞; and then, use L'Hospital's rule.

Indeterminate Powers 0^0, 1^∞ and ∞^0:

$\rightarrow$ We use a logarithmic "trick" as explained on the examples in these notes.

Attendance Popper#

Q# Compute $\lim_{x\to 0}\dfrac{\cos(x)-\cos(3x)}{\sin^2(x)}$.

a. 1 b. -1 c. –4 d. 4 e. DNE

Q# **Find:** $\lim_{x\to\infty}\left(1-\frac{3}{x}\right)^{2x} =$

A. 1 B. -6e C. e D. e^{-6} E. $3e^{-2}$

Q# Find $\lim_{x\to\infty}\frac{x}{\ln x}$.

a. 1 b. 0 c. DNE d. –2 e. None of these

Q# Compute $\lim_{x\to 0}\left(\frac{x-\sin x}{x-\tan x}\right)$.

a. ½ b. –1/2 c. 1 d. 0 e. DNE

Q# Compute $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$.

a. 1 b. 0 c. 3 d. 2 e. DNE

Q# Compute $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{x}{x-1}\right)$.

a. ½ b. – ½ c. 2 d. –2 e. DNE