Math 2413- Calculus I

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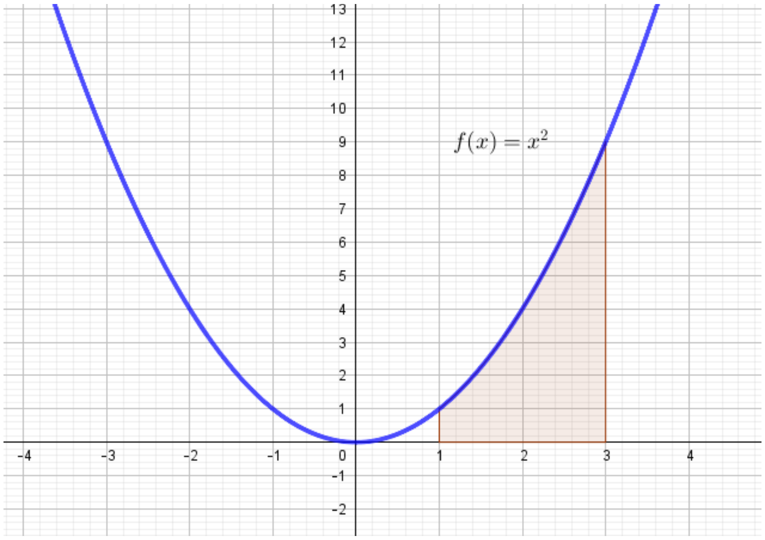
- Check CASA calendar for due dates.
- Bring "blank notes" to class. Completed notes will be posted after class.
- Do your best to attend every lecture and lab.
- Study after every lecture; work on the quiz covering the topic we cover on the lecture immediately afterwards. Retake your quizzes for more practice.
- Get help when you need help; bring your questions to the labs, or my office hours. We also have tutoring options on campus.
- Respect your friends in class; stay away from distractive behavior. Do your best to concentrate on the lecture.
- **If you email me, mention the course code in the subject line. Email is the best way to communicate with me outside of class. Teams chat messages are not monitored or replied to**.

Chapter 6 – Integration

Section 6.1 – Definite Integral

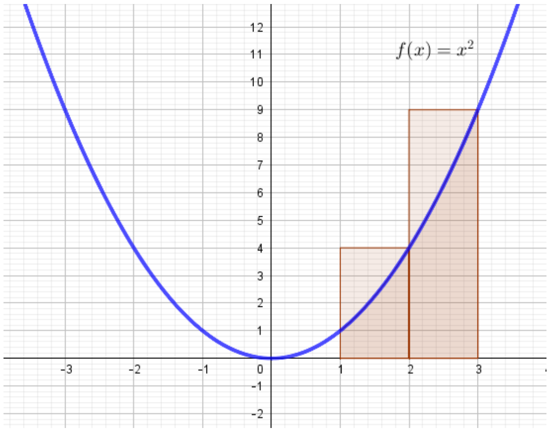
How could we find the area under the curve of $f(x) = x^2$ and above the x-axis for

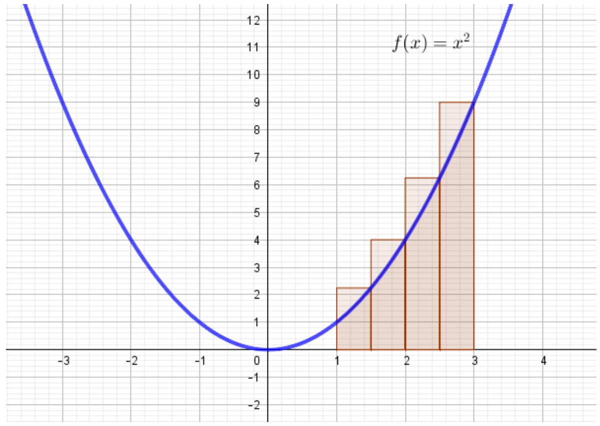
$x \in [1,3]$?

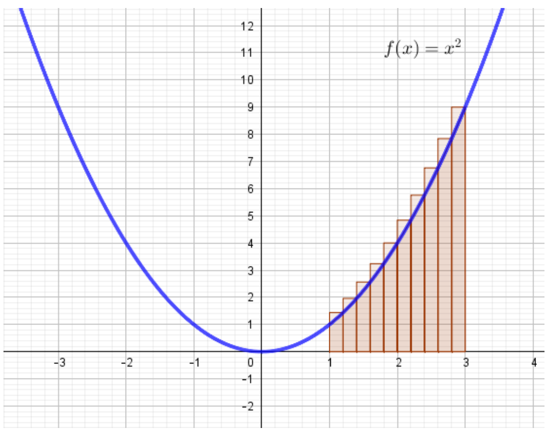


We will draw rectangles and use the sum of the areas of these rectangles to approximate the shaded area.

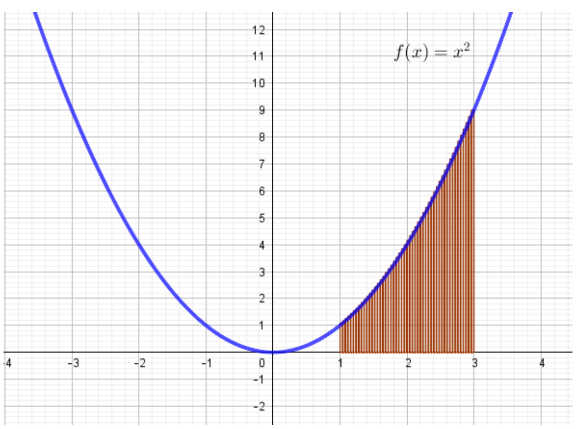
The more rectangles we make, the more accurate the area!







10 rectangles



50 rectangles.

Definition: A partition of a closed interval $[a,b]$ is a finite subset of $[a,b]$ that contains the points a and b.

When we say $\{x_0, x_1, ..., x_n\}$ is a partition of $[a,b]$, we imply that $a = x_0 < x_1 < ... < x_n = b$.

For example, the sets $P_1 = \{0,1\}$, $P_2 = \left\{0, \frac{1}{2}, 1\right\}$, $P_3 = \left\{0, \frac{1}{10}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{7}{8}, 1\right\}$ are partitions of the interval $[0,1]$.

If $P = \{x_0, x_1, ..., x_n\}$ is a partition of $[a,b]$, then P breaks the interval to subintervals:

$$[x_0, x_1], [x_1, x_2], [x_2, x_3], ..., [x_{n-1}, x_n] \quad \text{of lengths} \quad \Delta x_1, \Delta x_2, \Delta x_3, ..., \Delta x_n .$$

The lengths of these subintervals may or may not be equal.

If the lengths are equal, it is called a "**regular partition**" and $\Delta x = \dfrac{b-a}{n}$.

Using Riemann Sums –

Pick ***any*** point x_i* in the interval $[x_{i-1}, x_i]$.

This point x_i* may be

- the left endpoint of the interval $[x_{i-1}, x_i]$,
- the right endpoint of the interval $[x_{i-1}, x_i]$,
- the midpoint of the interval $[x_{i-1}, x_i]$,
- any other point in the interval $[x_{i-1}, x_i]$.

Now, using each point x_i*, form the products:

$$f(x_1*)\Delta x_1, \quad f(x_2*)\Delta x_2, ..., f(x_n*)\Delta x_n$$

The sum

$$S^*(P) = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \ldots + f(x_n^*)\Delta x_n$$

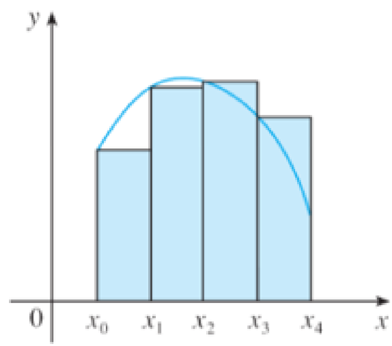
is called a **Riemann Sum.**

Most common approaches:

Left Hand Endpoint (LHE) Method:

If x_i^* is chosen to be *the left endpoint of the interval*, then $x_i^* = x_{i-1}$ and we have

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

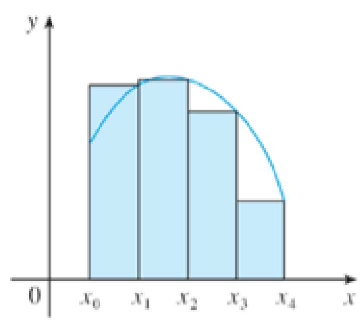


(a) Left endpoint approximation

Right Hand Endpoint (RHE) Method:

If x_i^* is chosen to be *the right endpoint of the interval*, then $x_i^* = x_i$ and we have

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$



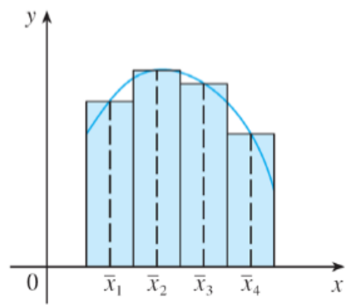
(b) Right endpoint approximation

Midpoint Method:

If x_i^* is chosen to be *midpoint of* i_{th} *subinterval* $[x_{i-1}, x_i]$,

say $\overline{x}_i = \frac{x_{i-1}+x_i}{2}$ and we have

$$M_n = \sum_{i=1}^{n} f(\overline{x}_i)\Delta x$$



(c) Midpoint approximation

General Formulas:

Left Hand Endpoint Method:

$$L_n = \frac{b-a}{n}\left[f\left(x_0\right) + f\left(x_1\right) + \cdots + f\left(x_{n-1}\right)\right]$$

Right Hand Endpoint Method:

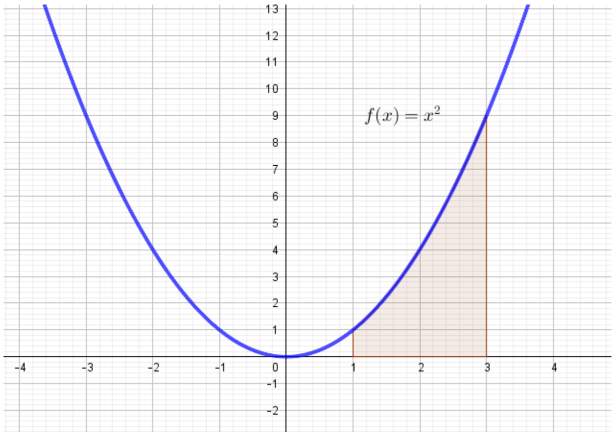
$$R_n = \frac{b-a}{n}\left[f\left(x_1\right) + f\left(x_2\right) + \cdots + f\left(x_n\right)\right]$$

Midpoint Method:

$$M_n = \frac{b-a}{n}\left[f\left(\frac{x_0 + x_1}{2}\right) + \cdots + f\left(\frac{x_{n-1} + x_n}{2}\right)\right]$$

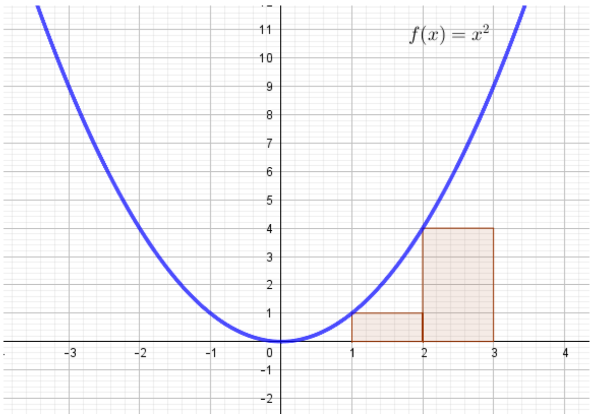
Using Riemann Sums to APPROXIMATE area under a curve

How can we APPROXIMATE the area under the curve of $f(x)=x^2$ and above the x-axis for $x \in [1,3]$?



Method 1: Use ***Left endpoint Method*** to approximate the area under the curve $f(x) = x^2$, over [1,3], use $n = 2$.

The interval [1,3] is split up into 2 pieces; each with width 1 unit; the subintervals are [1,2] and [2,3].



Because of the approach we are using, each rectangle touches the curve at the left endpoint. The heights are determined by the **left endpoint** of each subinterval. There are two rectangles; first with "height" $f(1)$; the second one with height $f(2)$. For each rectangle, the "width" is $\Delta x = \frac{b-a}{n} = \frac{3-1}{2} = 1$ unit.

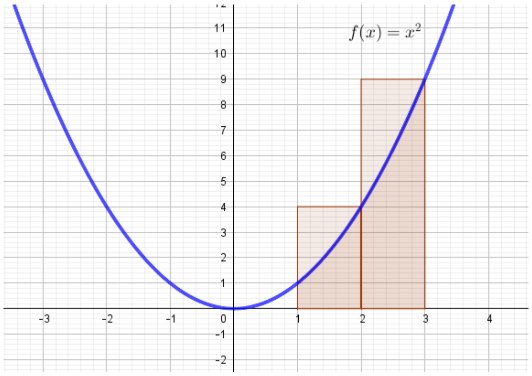
$$L_2 = \sum_{i=1}^{2} f(x_{i-1})\Delta x = \Delta x\left[f(x_0) + f(x_1)\right] = 1\left[f(1) + f(2)\right] = 1\left[1^2 + 2^2\right] = 5$$

Our approximation of the area using left endpoint approach with n=2 is 5.

Question: Is our approximation more than or less than the actual area?

Method 2: Use ***Right endpoint Method*** to approximate the area under the curve $f(x) = x^2$, over [1,3], use $n = 2$.

The interval [1,3] is split up into 2 pieces; each with width 1 unit; the subintervals are [1,2] and [2,3].



There are two rectangles; first with "height" f(2), the second one with height f(3). Because of the approach we are using, each rectangle touches the curve on the right endpoint, and hence the heights are determined by the **right endpoint** of each subinterval. For each rectangle, the "width" is $\Delta x = \frac{b-a}{n} = \frac{3-1}{2} = 1$ unit.

$$R_2 = \sum_{i=1}^{2} f(x_i)\Delta x = \Delta x\left[f(x_1) + f(x_2)\right] = 1\left[f(2) + f(3)\right] = 1\left[2^2 + 3^2\right] = 13$$

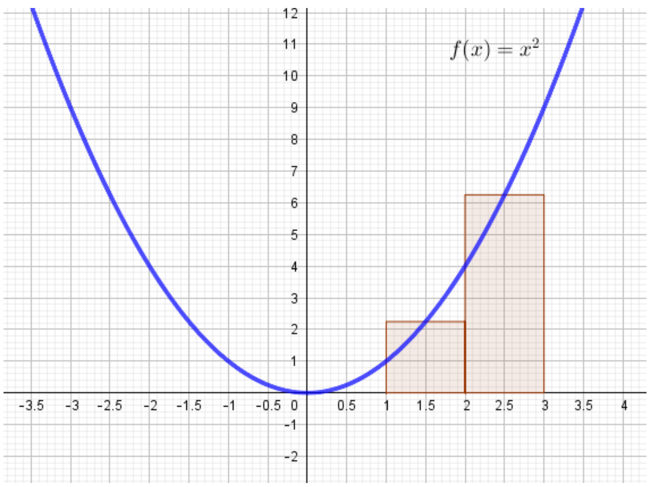
Our approximation of the area using right endpoint approach with n=2 is 13.

Question: Is our approximation more than or less than the actual area?

Method 3: Use ***Midpoint Method*** to approximate the area under $f(x) = x^2$, over [1,3], use $n = 2$.

The interval [1,3] is split up into 2 pieces; each with width 1 unit; the subintervals are [1,2] and [2,3].

For this approach, we need to determine the "midpoint" of each subinterval; mark 1.5 as the midpoint of [1,2] and mark 2.5 as the midpoint of [2,3].



For this method, the rectangles touch the curve at the midpoint. The height of each rectangle is determined by $f(\text{midpoint})$. First rectangle has height $f(1.5)$ and the second rectangle has height $f(2.5)$. Again, the width is 1.

$$M_2 = \sum_{i=1}^{2} f(\overline{x_i})\Delta x = \Delta x\left[f(1.5) + f(2.5)\right] = 1\left[(1.5)^2 + (2.5)^2\right] = 2.25 + 6.25 = 8.5$$

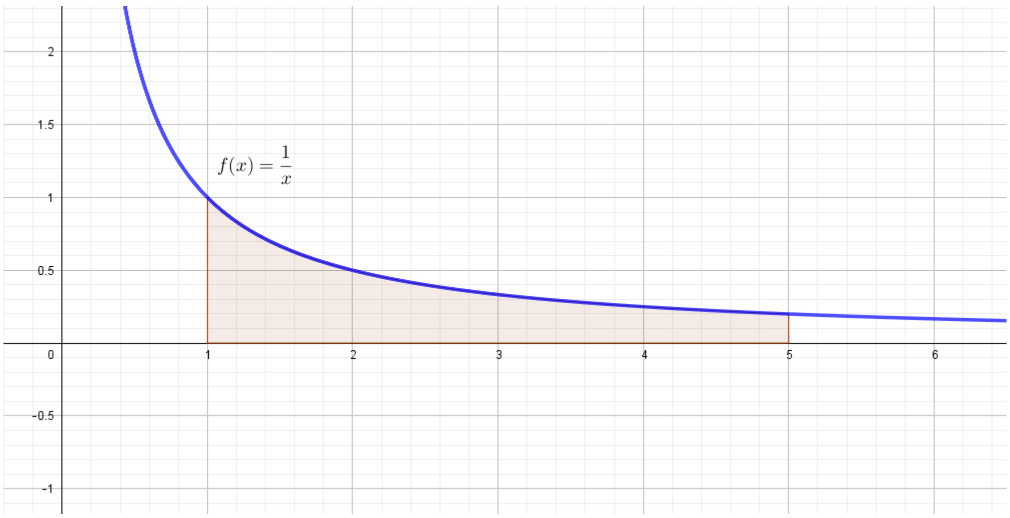
Our approximation of the area using midpoint approach with n=2 is 8.5.

Question: Is our approximation more than or less than the actual area?

Question: Which one is closer to the exact area? Which one is a better approximation?

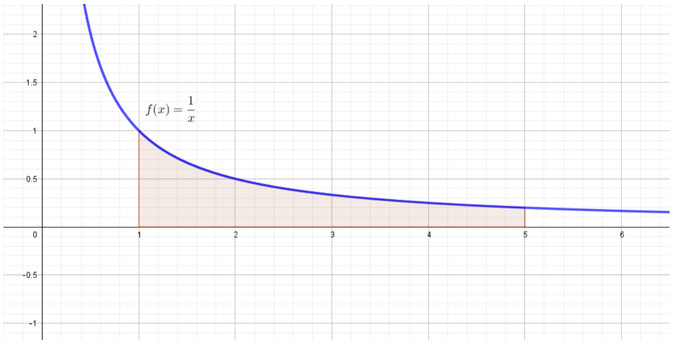
Our next example is about the following curve:

The area under the curve $f\left(x\right)=\dfrac{1}{x}$ over the interval [1,5] is shaded below:



For each problem, APPROXIMATE the area under the curve using the given number and type of Riemann sums.

Example 1: APPROXIMATE the area under the curve $f(x)=\frac{1}{x}$, over the interval: [1,5] using the given type of Riemann sums:



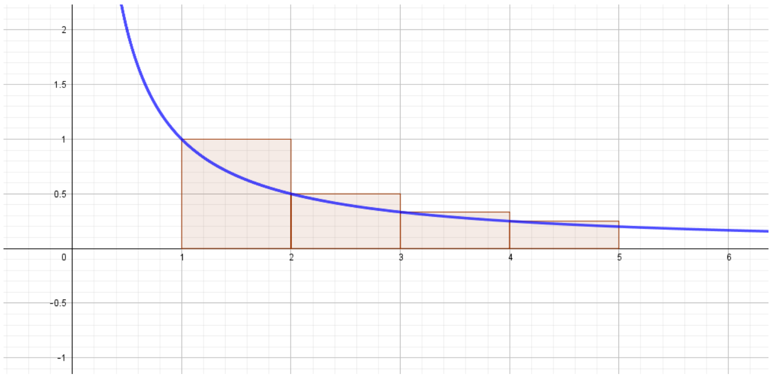
(a) Use: **Left endpoint Method** with $n=4$.

The function: $f(x)=\frac{1}{x}$, over interval: [1,5].

The interval [1,5] is divided into 4 subintervals of equal width; $\Delta x=\frac{b-a}{n}=\frac{5-1}{4}=1$.

The subintervals are [1,2], [2,3], [3,4], and [4,5]. Draw a rectangle for each subinterval. Left end point approach instructs us that the rectangles should touch the curve on the left endpoint of the subinterval.

- The rectangle over [1,2] should touch the curve at x=1 (left endpoint of this subinterval); hence the height of that rectangle is determined by $f(1)$.
- The rectangle over [2,3] should touch the curve at x=2 (left endpoint of the subinterval); hence the height of that rectangle is determined by $f(2)$, and so on.
- Note that the 4^{th} rectangle will cover the subinterval [4,5] and we will use x=4 as the left endpoint.



Now, add the areas of all 4 rectangles:

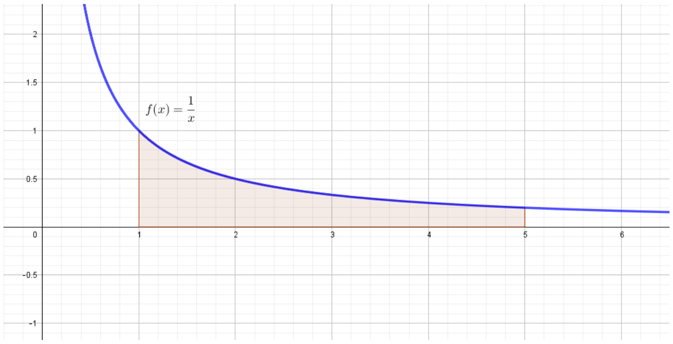
$$L_4 = \sum_{i=1}^{4} f(x_{i-1})\Delta x = \Delta x\left[f(x_0) + f(x_1) + f(x_2) + f(x_3)\right] = 1\cdot\left[f(1) + f(2) + f(3) + f(4)\right]$$

$$= 1\cdot\left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right] = \frac{25}{12}$$

Our approximation of the area under the curve $f\left(x\right) = \frac{1}{x}$, over the interval: [1,5] using left endpoint approach with n=4 is $L_4 = \frac{25}{12} = 2.08$.

Question: Is our approximation more than or less than the actual area?

Example 1: APPROXIMATE the area under the curve $f(x) = \frac{1}{x}$, over the interval: [1,5] using the given type of Riemann sums:



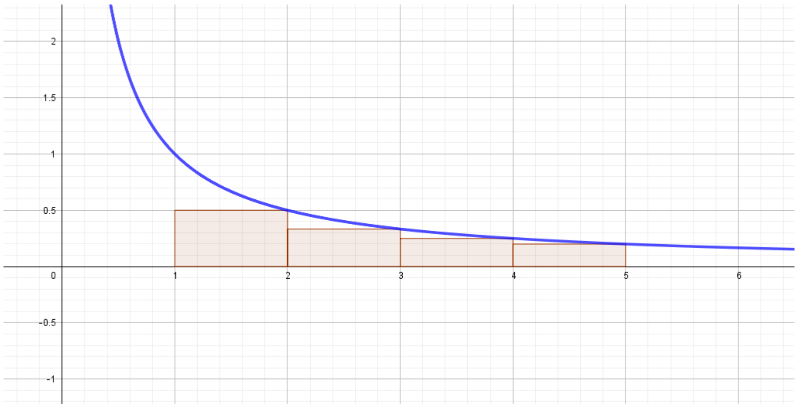
(b) Use: **Right Endpoint Method** with n=4.

Function: $f(x) = \frac{1}{x}$, interval: [1,5].

The interval [1,5] is divided into 4 subintervals of equal width; $\Delta x = \frac{b-a}{n} = \frac{5-1}{4} = 1$.

The subintervals are [1,2], [2,3], [3,4], and [4,5]. Draw a rectangle for each subinterval. Left end point approach instructs us that the rectangles should touch the curve on the left endpoint of the subinterval.

- The rectangle over [1,2] should touch the curve at x=2 (right endpoint of this subinterval); hence the height of that rectangle is determined by $f(2)$.
- The rectangle over [2,3] should touch the curve at x=3 (left endpoint of the subinterval); hence the height of that rectangle is determined by $f(3)$, and so on.
- Note that the 4^{th} rectangle will cover the subinterval [4,5] and we will use x=5 as the right endpoint.



Now, add the areas of all 4 rectangles:

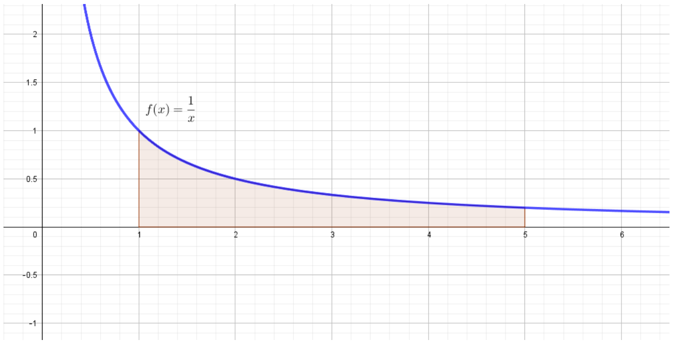
$$R_4 = \sum_{i=1}^{4} f(x_i)\Delta x = \Delta x\left[f(x_1) + f(x_2) + f(x_3) + f(x_4)\right] = 1\cdot\left[f(2) + f(3) + f(4) + f(5)\right]$$

$$= 1\cdot\left[\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right] = \frac{77}{60}$$

Our approximation of the area under the curve $f\left(x\right)=\frac{1}{x}$, over the interval: [1,5] using right endpoint approach with n=4 is $R_4 = \frac{77}{60} = 1.28$.

Question: Is our approximation more than or less than the actual area?

Example 1: APPROXIMATE the area under the curve $f(x)=\frac{1}{x}$, over the interval: [1,5] using the given type of Riemann sums:



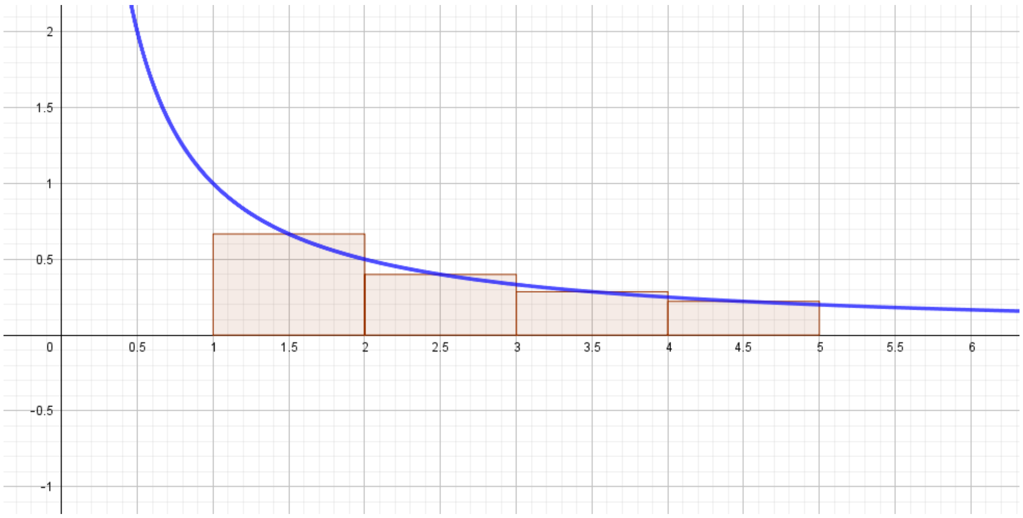
(c) Use: **MIDPOINT Method** with n=4.

Function: $f(x)=\frac{1}{x}$, interval: [1,5].

The interval [1,5] is divided into 4 subintervals of equal width; $\Delta x=\frac{b-a}{n}=\frac{5-1}{4}=1$.

The subintervals are [1,2], [2,3], [3,4], and [4,5]. Determine the midpoint of each subinterval; 1.5, 2.5, 3.5, and 4.5. Draw a rectangle for each subinterval that will touch the curve at the midpoint.

- The rectangle over [1,2] should touch the curve at x=1.5 (midpoint of this subinterval); hence the height of that rectangle is determined by $f(1.5)$.
- The rectangle over [2,3] should touch the curve at x=2.5 (midpoint of the subinterval); hence the height of that rectangle is determined by $f(2.5)$, and so on.



$$M_4 = \sum_{i=1}^{4} f(\overline{x_i})\Delta x = 1\cdot\left[f(1.5) + f(2.5) + f(3.5) + f(4.5)\right]$$

$$= 1\cdot\left[\frac{1}{1.5} + \frac{1}{2.5} + \frac{1}{3.5} + \frac{1}{4.5}\right] = \frac{1488}{945} \overset{calculator}{=} 1.58$$

Our approximation of the area under the curve $f(x) = \frac{1}{x}$, over the interval: [1,5] using midpoint approach with n=4 is $M_4 = 1.58$.

Question: Is our approximation more than or less than the actual area?

NOTE: A calculator may be needed for many problems on the online quiz covering this section. For these, you can use any simple calculator. If you wish, you can use GEOGEBRA as a free online tool. (On the exams, we will ask questions in a way that calculators will not be needed; for example, we might ask "comparison" questions.)

The commands for Geogebra are the following:

Step 1: Enter the function.

Step 2: For Riemann sums, the commands start with "RectangleSum".

RectangleSum(<Function>, <Start x-Value>, <End x-Value>, <Number of Rectangles>, <Position for rectangle start>)

Number of rectangles is the n value you want to use.

Position is: 0 for left end point, 1 for right end point, 0.5 for midpoint approaches.

For the interval [a,b], the Start Value is a, and value is b.

Example:

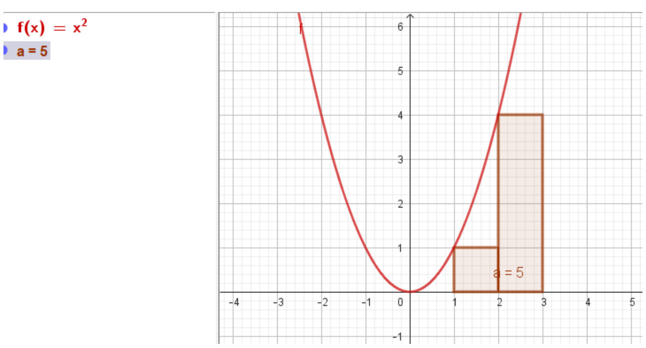
f(x)=x^2 over the interval [1,3], use n=2.

If the interval is [1,3]; start value is 1, end value is 3.

Input: f(x)=x^2

For the left endpoint approach: Input: RectangleSum(f(x), 1,3, 2, **0**)

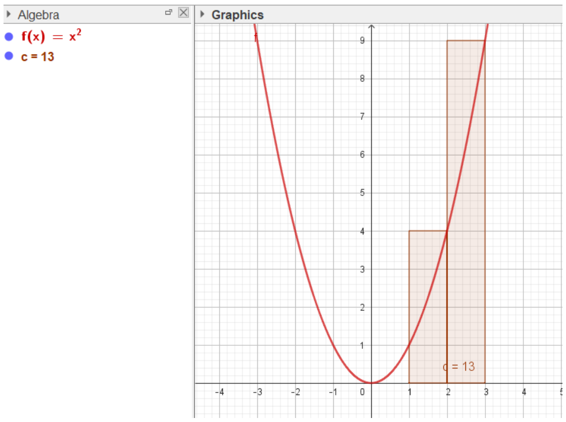
Enter: you get 5. That is, L_2 =5.



For the right end point approach:

Input: RectangleSum(f(x), 1, 3, 2, 1)

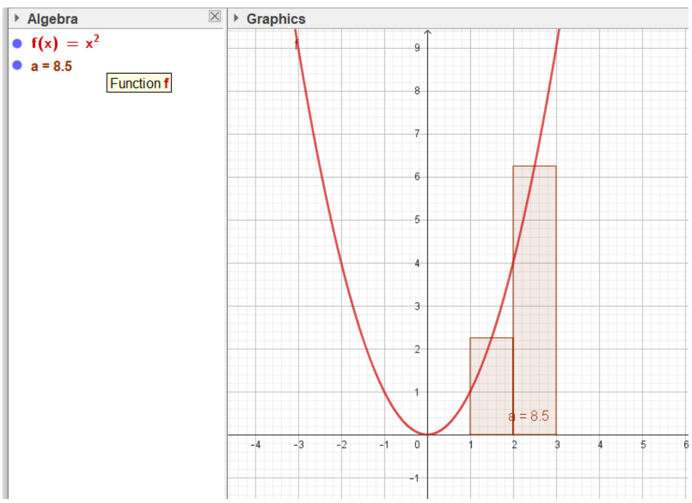
Answer: R_2= 13.



For the midpoint approach,

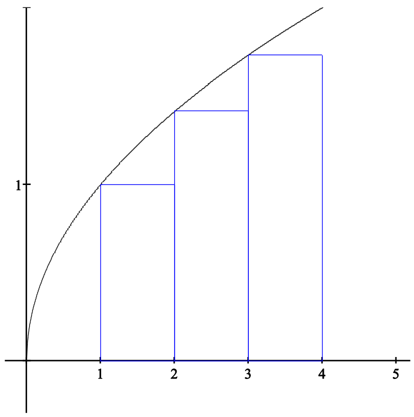
Input: RectangleSum(f(x), 1, 3, 2, 0.5); enter.

We get M_2 = 8.5.



Exercise: Approximate the area under the curve over the given interval using Riemann Sums.

1. Left endpoint $f(x) = \sqrt{x}$ $[1,4]$ $n = 3$

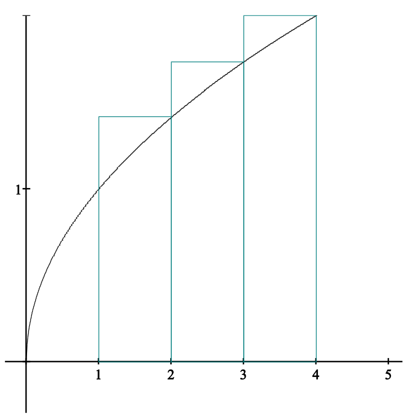


You can check your answer from Geogebra:

Type - input: f(x)=sqrt(x) (enter)

Input: RectangleSum(f(x), 1, 4, 3, 0) (enter)

2. Right endpoint $f(x) = \sqrt{x}$ $[1,4]$ $n = 3$

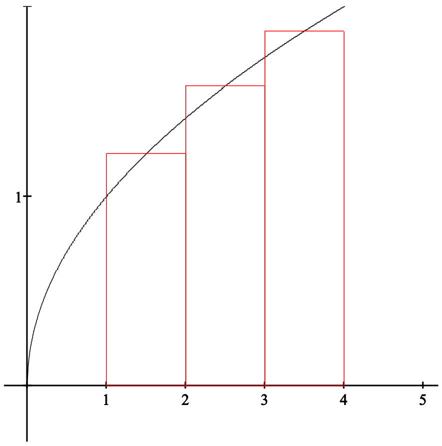


You can check your answer from Geogebra:

Type: f(x)=sqrt(x)

Input: RectangleSum(f(x), 1, 4, 3, 1)

3. Midpoint $f(x)=\sqrt{x}$ $[1,4]$ $n=3$



You can check your answer from Geogebra:

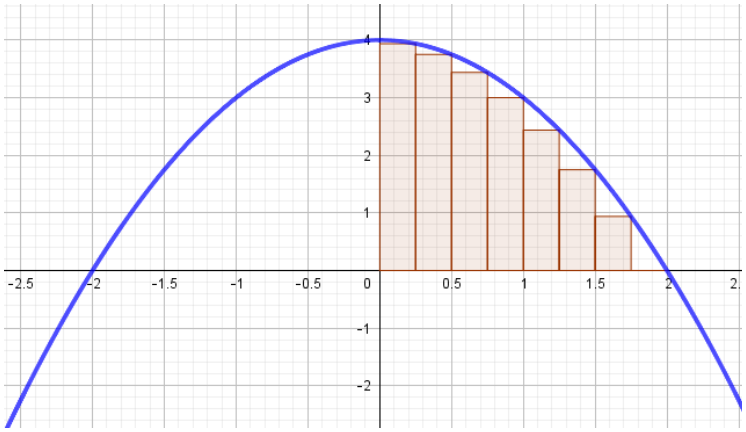
Type: f(x)=sqrt(x)

Input: RectangleSum(f(x), 1, 4, 3, 0.5)

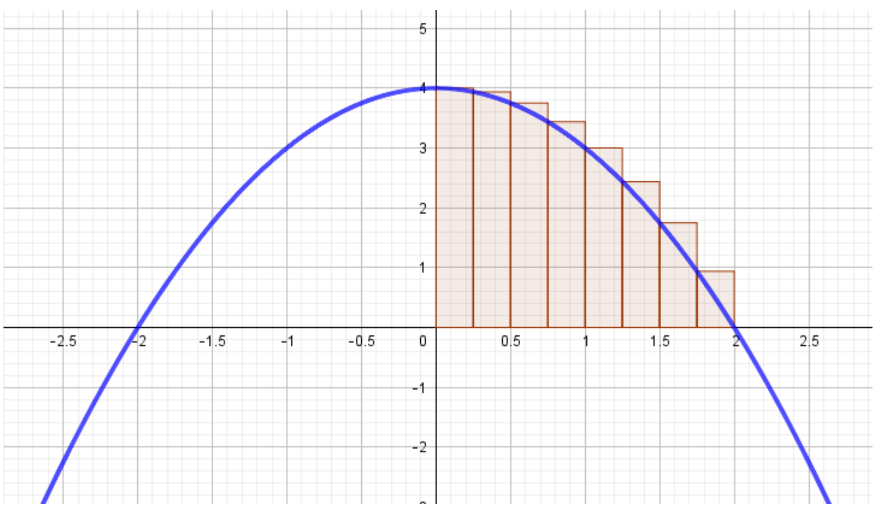
Exercise: Approximate the area under the curve over the given interval using Riemann Sums.

Left endpoint $f(x) = 4 - x^2 \quad , over: [0,2], \qquad use: \qquad n = 8$

Note: the width of each subinterval is 2/8=1/4.



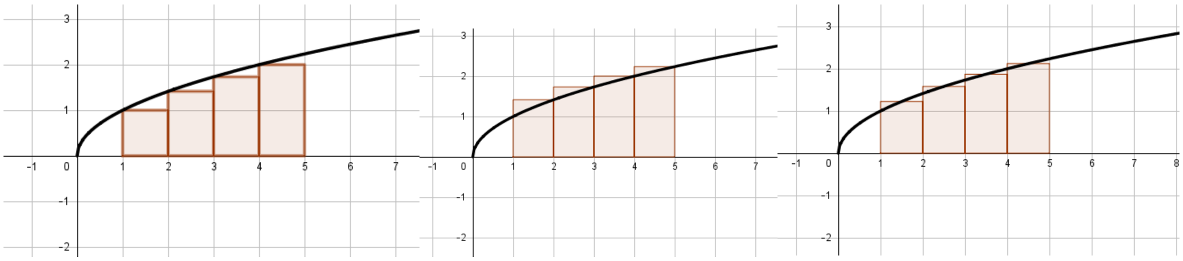
Right endpoint $f(x) = 4 - x^2 \quad , over: [0,2], \qquad use: \qquad n = 8$



Compare: L_8, R_8 and the actual area; order them from the biggest to the smallest.

REMARK: You can compare these Reimann sums without actually computing them (important for Calc 2). Study the examples below.

(i) Compare $\int_1^5 f(x)dx$ and L_4, R_4, M_4

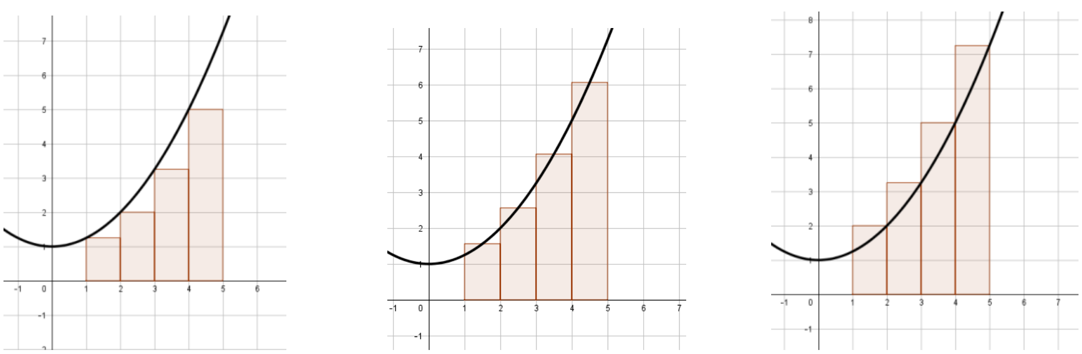


The function is increasing,

$L_4 < M_4 < R_4$ $\qquad$ $L_4 < \int_1^5 f(x)dx < R_4$

We can't compare M_4 and the integral without computations.

(ii) Compare $\int_1^5 f(x)dx$ and L_4, R_4, M_4

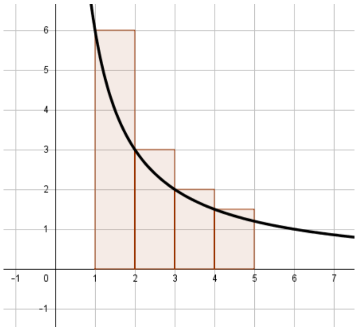
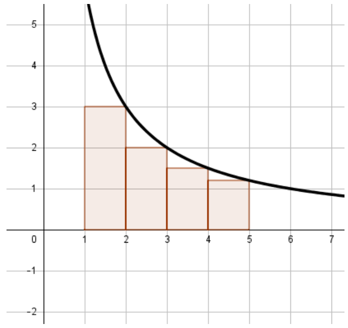
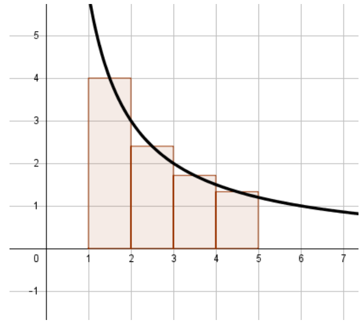


The function is increasing,

$L_4 < M_4 < R_4$ $\qquad$ $L_4 < \int_1^5 f(x)dx < R_4$

We can't compare M_4 and the integral without computations.

(iii) Compare $\int_1^5 f(x)dx$ and L_4, R_4, M_4

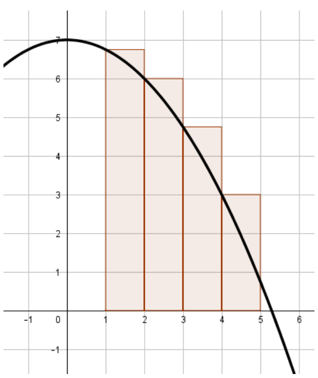
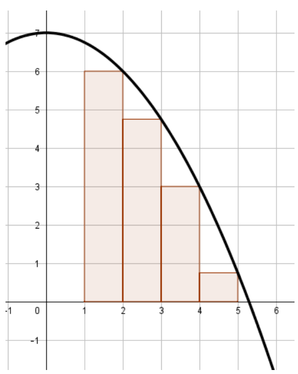
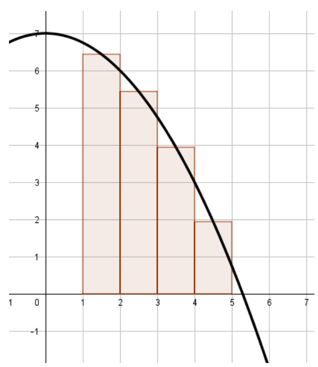
The function is <u>decreasing</u>,

$R_4 < M_4 < L_4$

$R_4 < \int_1^5 f(x)dx < L_4$

We can't compare M_4 and the integral without computations.

(iv) Compare $\int_1^5 f(x)dx$ and L_4, R_4, M_4

The function is <u>decreasing</u>,

$R_4 < M_4 < L_4$

$R_4 < \int_1^5 f(x)dx < L_4$

We can't compare M_4 and the integral without computations.

Upper Sum – Lower Sum

Definition: Let f be a continuous function on $[a,b]$ and $P=\{x_0,x_1,...,x_n\}$ be a partition of $[a,b]$.

The sum:

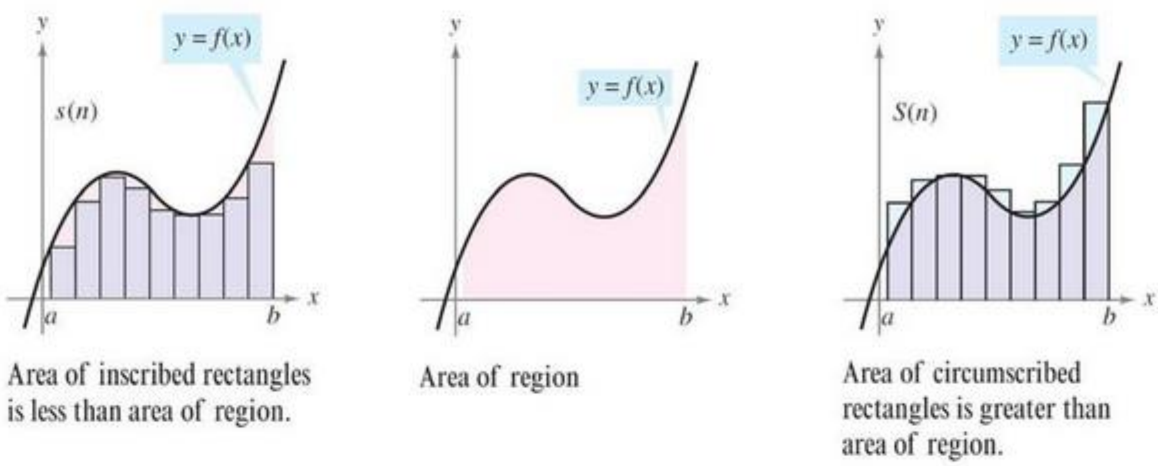
$$U_f(P)=M_1\Delta x_1+M_2\Delta x_2+M_3\Delta x_3+...+M_n\Delta x_n$$

is called the **upper sum of** f with respect to the partition P.

The sum:

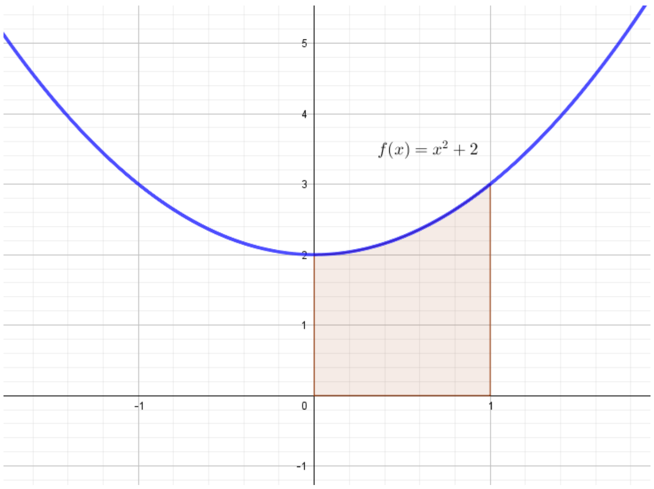
$$L_f(P)=m_1\Delta x_1+m_2\Delta x_2+m_3\Delta x_3+...+m_n\Delta x_n$$

is called the **lower sum of** f with respect to the partition P.

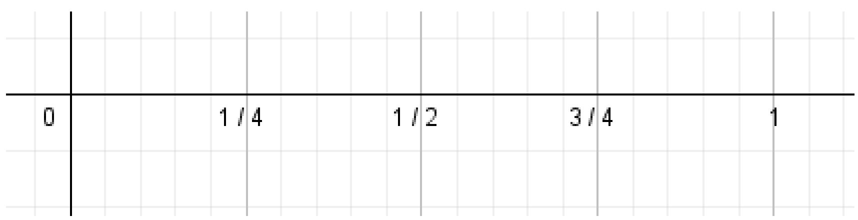


Lower sum is always less than equal to the area; Uppersum is always greater than or equal to the area (if the function is nonnegative).

Example: Find an **Upper Sum** for $f(x) = x^2 + 2$, over the interval $[0,1]$ if the partition is $P = \left\{0, \frac{1}{4}, \frac{1}{2}, 1\right\}$



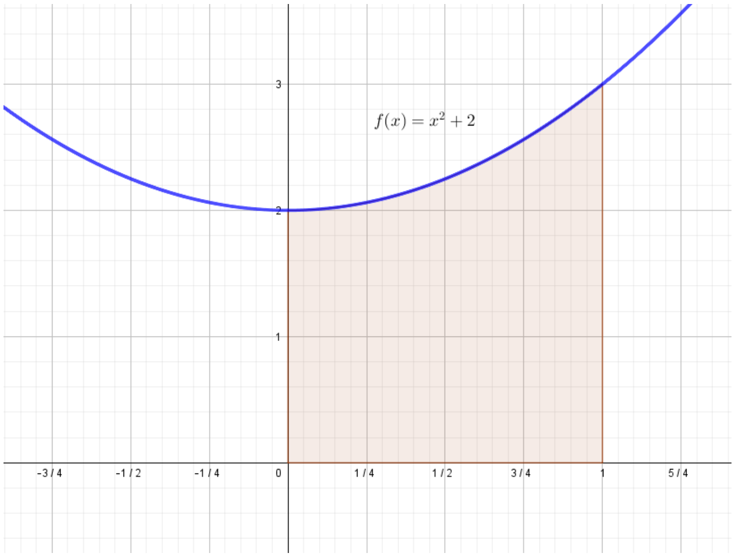
Given partition: $P = \left\{0, \frac{1}{4}, \frac{1}{2}, 1\right\}$, the interval [0,1] is divided into the following 3 subintervals:



$\left[0, \frac{1}{4}\right]$; the width is ¼.

$\left[\frac{1}{4}, \frac{1}{2}\right]$; the width is ¼.

$\left[\frac{1}{2}, 1\right]$; the width is ½.

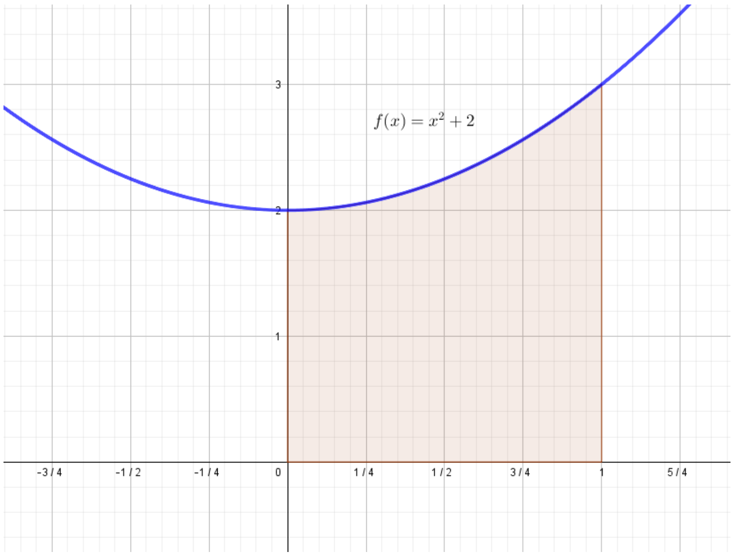


$$U_f(P) = M_1\Delta x_1 + M_2\Delta x_2 + M_3\Delta x_3 = f\left(\frac{1}{4}\right)\cdot\frac{1}{4} + f\left(\frac{1}{2}\right)\cdot\frac{1}{4} + f(1)\cdot\frac{1}{2}$$

$$= \left(\frac{1}{4^2} + 2\right)\cdot\frac{1}{4} + \left(\frac{1}{2^2} + 2\right)\cdot\frac{1}{4} + \left(1^2 + 2\right)\cdot\frac{1}{2}$$

$$= \frac{165}{64} = 2.58$$

Example: Find a **Lower Sum** for $f(x) = x^2 + 2$, over the interval $[0,1]$ if the partition is $P = \left\{0, \frac{1}{4}, \frac{1}{2}, 1\right\}$



$$L_f(P) = m_1\Delta x_1 + m_2\Delta x_2 + m_3\Delta x_3 = f(0)\cdot\frac{1}{4} + f\left(\frac{1}{4}\right)\cdot\frac{1}{4} + f\left(\frac{1}{2}\right)\cdot\frac{1}{2}$$

$$= (0+2)\cdot\frac{1}{4} + \left(\frac{1}{4^2} + 2\right)\cdot\frac{1}{4} + \left(\frac{1}{2^2} + 2\right)\cdot\frac{1}{2}$$

$$= \frac{137}{64} = 2.14$$

Most of the time, we work with subintervals of equal width instead of a given partition (as in the previous methods).

Let's use Geogebra on such an example:

Commands:

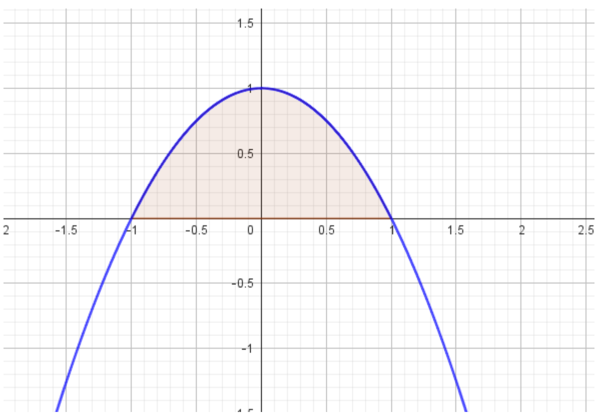
UpperSum(<Function>, <Start x-Value>, <End x-Value>, <Number of Rectangles>)

Or

LowerSum(<Function>, <Start x-Value>, <End x-Value>, <Number of Rectangles>)

Example: We want to approximate the area under the curve over the interval [-1,1] using upper and lower Riemann Sums:

$f(x) = 1 - x^2$, for $x \in [-1,1]$; use n=4, with equal width.



Upper Sum:

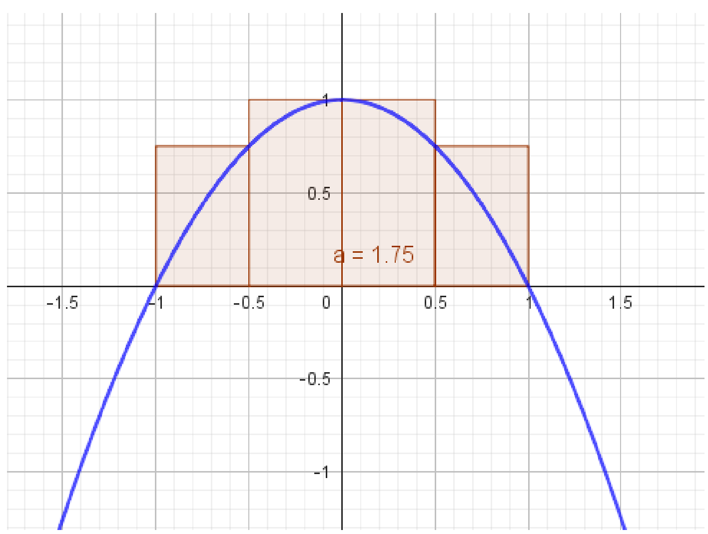
Geogebra commands:

Input: f(x)= 1 -x^2 (enter)

Input: UpperSum(f(x), -1, 1, 4) (enter)

We get:

Uppersum=1.75.



Lower Sum:

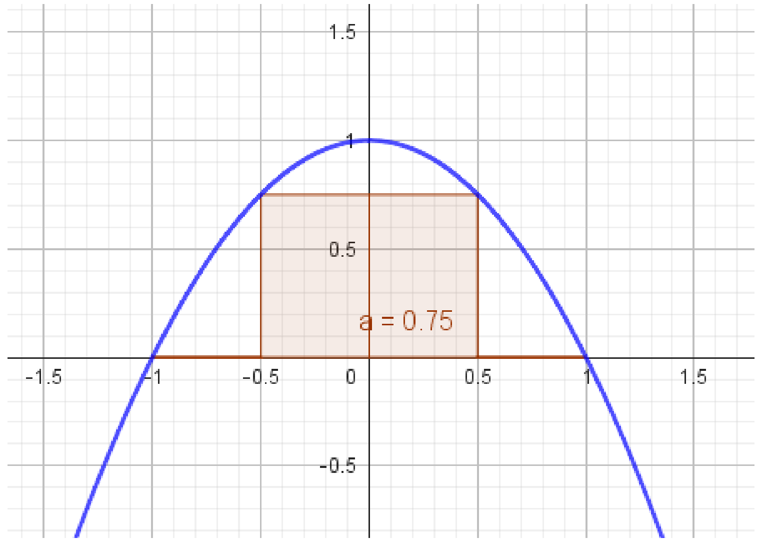
Geogebra commands:

Input: f(x)= 1 -x^2 (enter)

Input: LowerSum(f(x), -1, 1, 4) (enter)

We get:

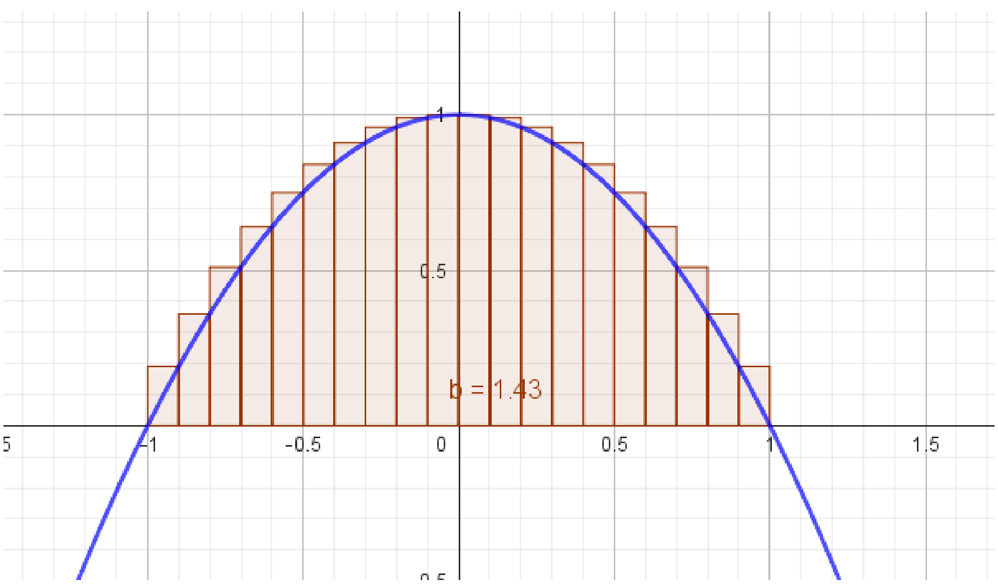
LowerSum=1.75.



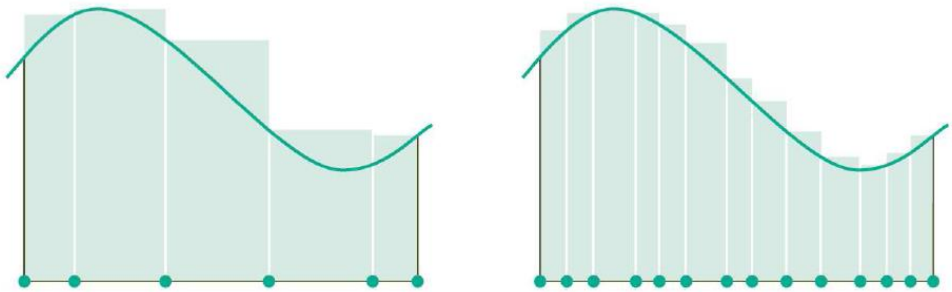
Note that since the minimum value is used as Rectangle “heights”, we have 2 rectangles with “height” 0. This is possible.

Note: By changing the value of n on Geogebra, you can see how these approximations get better; the error gets smaller and smaller.

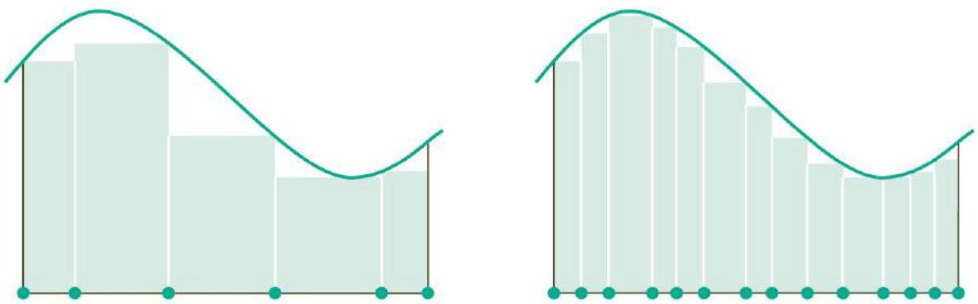
UpperSum with n=20: Input: UpperSum(f(x),-1,1,20)



More on upper sum and lower sum:



as points are added to a partition, the upper sums tend to get smaller



as points are added to a partition, the lower sums tend to get bigger

For a function f which is continuous on $[a,b]$, there is one and only one number that satisfies the inequality

$$L_f(P) \le I \le U_f(P), \text{ for all partitions } P \text{ of } [a,b].$$

And that number is the number we use to define the definite integral.

DEFINITE INTEGRAL

Definition: Let f be continuous on $[a,b]$. The unique number that satisfies

$$L_f(P) \le I \le U_f(P), \text{ for all partitions } P \text{ of } [a,b]$$

is called the **definite integral** of f from a to b and is denoted by:

$$\int_a^b f(x)dx.$$

We read $\int_a^b f(x)dx$ as: "the integral from a to b of $f(x)$ with respect to x".

The component parts have these names:

$\int$: the integral sign

a : lower limit of integration

b: upper limit of integration

$f(x)$: integrand.

The procedure of calculating the integral is called **integration**.

Notice that the" dx" is a part of the integral notation and it indicates the independent variable in discussion;

$\int_a^b f(x)dx$: the variable is x,

$\int_a^b f(u)du$: the variable is u,

$\int_a^b g(t)dt$: the variable is t.

Ex: $\int_0^1 x^2 dx$

The function we are integrating is $f(x) = x^2$; we are integrating over the interval $[0,1]$. The integral is with respect to the variable x.

The answer is a real number representing the area under the curve $f(x) = x^2$ over $[0,1]$ (since the function is non-negative).

Ex: $\int_{-1}^{2} \sqrt{2+t}\ dt$

The function we are integrating is $g(t) = \sqrt{2+t}$; we are integrating over the interval $[-1,2]$. The variable is t ; the integral is with respect to t.

Ex: $\int_2^5 \frac{1}{u}\ du$

The function we are integrating is $h(u) = \frac{1}{u}$; we are integrating over the interval $[2,5]$. The variable is u ; the integral is with respect to u.

Properties of Definite Integral

When we defined the definite integral $\int_a^b f(x)\,dx$, we assumed that $a < b$. However, the integral makes sense even if $a > b$. We can integrate from right to left by defining

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

Moreover, if $a = b$, then the integral is defined to be 0: $\int_a^a f(x)\,dx = 0$.

Assume that f and g are continuous functions.

READ THESE PROPERTIES!

Properties:

1) $\int_a^b k \; dx = k(b-a)$, where k is a constant number.

2) $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$.

3) $\int_a^b kf(x)dx = k\int_a^b f(x)dx$, where k is a constant number.

Ex: $\int_0^1 5x^2dx = 5\int_0^1 x^2dx$

4) If $f(x) \geq 0$ over $[a,b]$, then $\int_a^b f(x)dx \geq 0$.

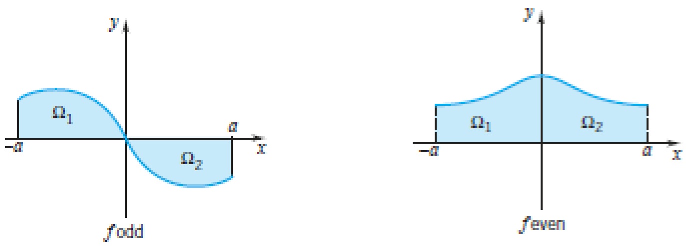
5) If $f(x) \geq g(x)$ over $[a,b]$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

6) If $m \le f(x) \le M$ over $[a,b]$, then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$.

7) $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$.

8) If f is an ODD function, then $\int_{-a}^{a} f(x)\,dx = 0$.

If f is an EVEN function, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.



Ex: $f(x) = x^2$ is an EVEN function, so $\int_{-1}^{1} f(x)dx = 2\int_{0}^{1} f(x)dx$.

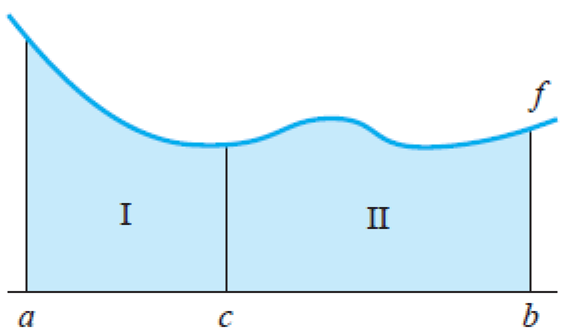
Ex: $g(x) = x^3$ is an ODD function, so $\int_{-1}^{1} g(x)dx = 0$.

Example: Use properties of integration to compute: $\int_{-\pi}^{\pi} \sin(x)\; dx$

Theorem 6.1.1: If f is continuous on $[a,b]$ and if $a<c<b$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

That is, we can split up an integral into several pieces. For nonnegative functions, this theorem can be easily understood using the area under the curve.



The theorem states that

Area of Region I + Area of Region II = Area under the curve from $x=a$ to $x=b$.

Example: $$\int_0^1 x^2dx + \int_1^4 x^2dx = \int_0^4 x^2dx$$

Example: If needed, $\int_1^{10} f(x)dx$ can be split up as: $\int_1^6 f(x)dx + \int_6^{10} f(x)dx$.

Example: Given

$$\int_1^4 f(x)\ dx = 9,\ \int_4^{10} f(x)\ dx = 12,\ \int_6^{10} f(x)\ dx = 7.$$

Evaluate the following integrals.

a) $\int_4^1 2f(x)\ dx$

b) $\int_1^{10} f(x)\ dx$

c) $\int_4^6 f(x)\ dx$

Example: Given

$$\int_1^5 f(x)\ dx = 10,\ \int_1^5 g(x)\ dx = 4.$$

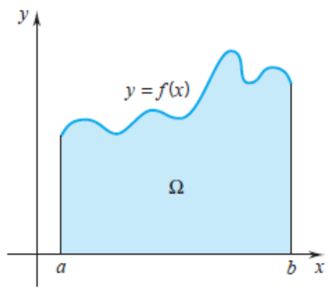
Evaluate the following integral:

$$\int_1^5 \left[f(x) - 2g(x) \right]\ dx$$

Area Under the Graph of a Nonnegative Function

Fact: If $y = f(x)$ is nonnegative and integrable over the interval $[a,b]$, then the **area under the curve** $y = f(x)$ **over** $[a,b]$ is:

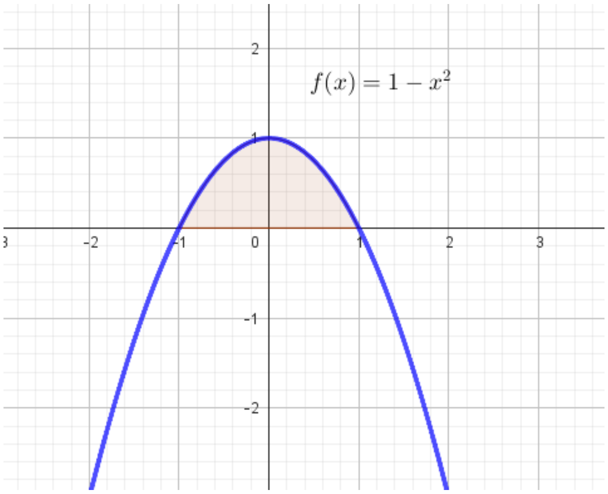
$$A = \int_a^b f(x)\,dx\,.$$



$$\text{area of } \Omega = \int_a^b f(x)\,dx.$$

Given integral, finding the area:

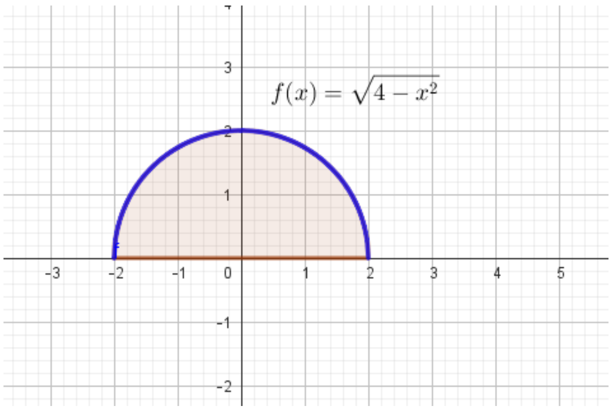
Example: **Given:** $\int_{-1}^{1}\left(1-x^2\right)=\frac{4}{3}$. Find the area between $f(x)=1-x^2$ and the x-axis over the interval $\left[-1,1\right]$. The region is shaded below.



Conclusion: The area of the shaded region is: $\frac{4}{3}$

Given area, finding the integral:

Example: The graph of $f(x)=\sqrt{4-x^2}$ is given below; the area between this function and the x-axis is shaded. The shaded area below traces a semicircle of radius 2; hence the area of the shaded region is 2π.



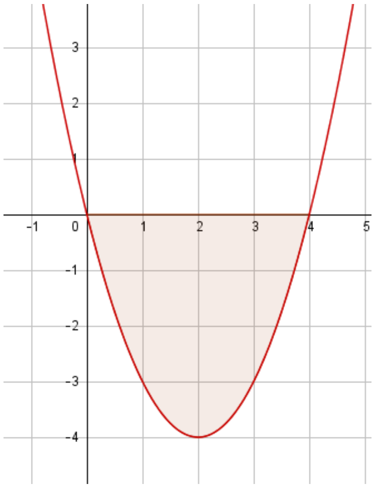
Conclusion: Since the function $f(x)=\sqrt{4-x^2}$ is above the x-axis, this means:

$$\int_{-2}^{2}\sqrt{4-x^2}\,dx = 2\pi \ .$$

What if $f(x)$ is BELOW the x-axis?

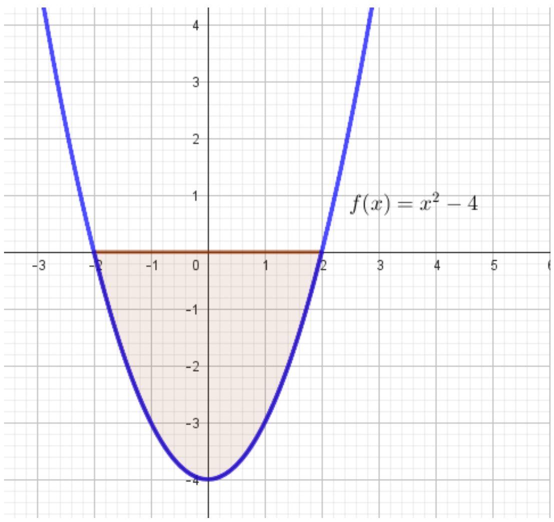
Then**, the function is negative, and hence, its integral is negative.**

But remember that "area" can NOT be negative:



Area between $f(x)$ and the x-axis on [0,4] is: $\left|\int_0^4 f(x)dx\right| = -\int_0^4 f(x)dx$

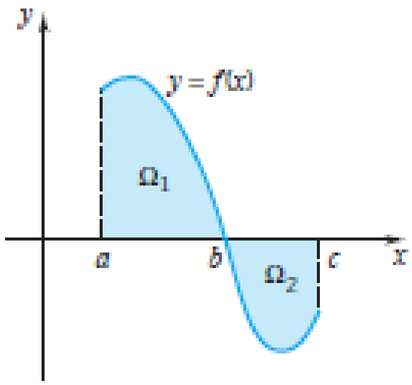
Example: Given: $\int_{-2}^{2}\left(x^2-4\right)=-\frac{32}{3}$. Find the area between the curve $f(x)=x^2-4$ and the x-axis over the interval $[-2,2]$; the area is shaded below.

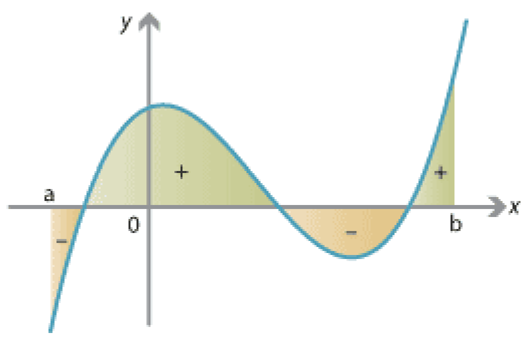


Conclusion: The area of the shaded region is: $\frac{32}{3}$

Remember: the definite integral of a function can be negative; the area of a region is never negative.

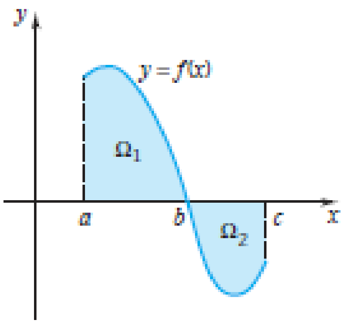
What if it is sometimes above and sometimes below the x-axis?





If the curve is sometimes negative, then one can split the region into pieces using the roots of the function as the limits of the integral.

Consider the function whose graph is given below:

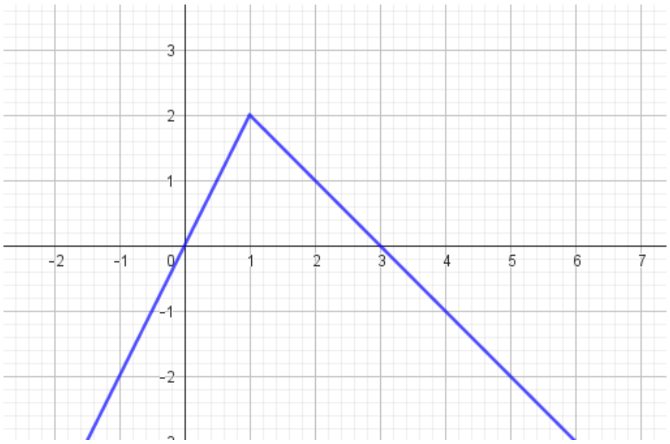


Here, Area of $\Omega_1 = \int_a^b f(x)\,dx$ and Area of $\Omega_2 = \left|\int_b^c f(x)\,dx\right| = -\int_b^c f(x)\,dx$

Total area = Area of Ω_1 + Area of $\Omega_2 = \int_a^b f(x)\,dx + \left(-\int_b^c f(x)\,dx\right)$

Hence, $\int_a^c f(x)\,dx$ **<u>is not equal</u>** to the total area under the curve from $x = a$ to $x = c$.

Example: The following graph belongs to f(x).



We can observe that the area under f(x) from 1 to 3 is: 2. (area of a right triangle: 2*2/2=2).

We conclude: $\int_1^3 f(x)\,dx = 2$ since the function is positive on that interval.

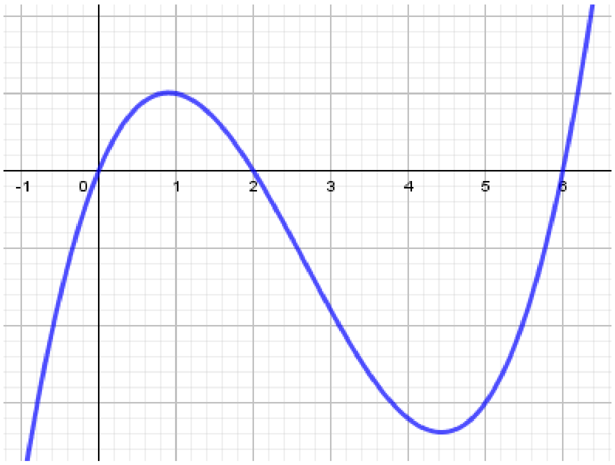
Area under f(x) from 3 to 5 is also 2.

We conclude: $\int_3^5 f(x)\,dx = -2$ since the function is negative on that interval.

If we want to compute $\int_1^5 f(x)\,dx$, we would get:

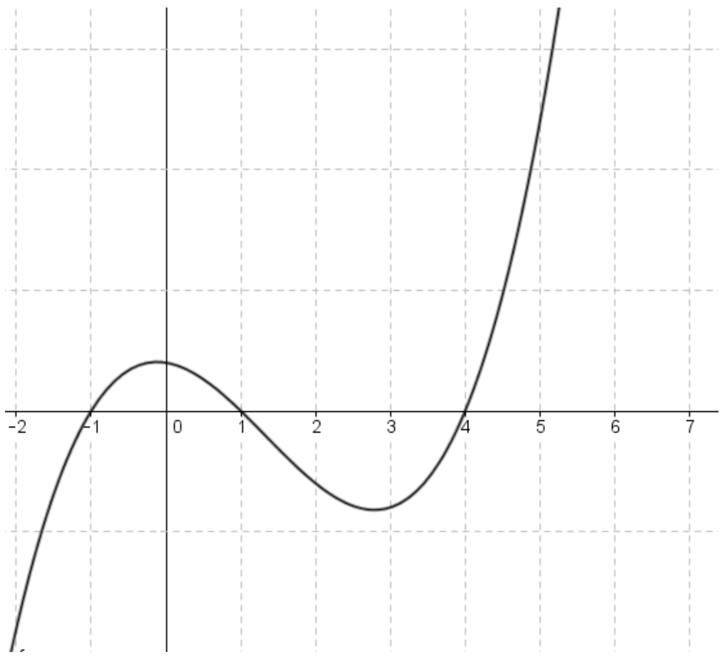
$$\int_1^5 f(x)\,dx = \int_1^3 f(x)\,dx + \int_3^5 f(x)\,dx = (2) + (-2) = 0 .$$

Example: The area between the function $f(x)$ and the x-axis over the interval [0,2] is 5, and the area over the interval [2,6] is 14. Find: $\int_0^6 f(x)dx$.

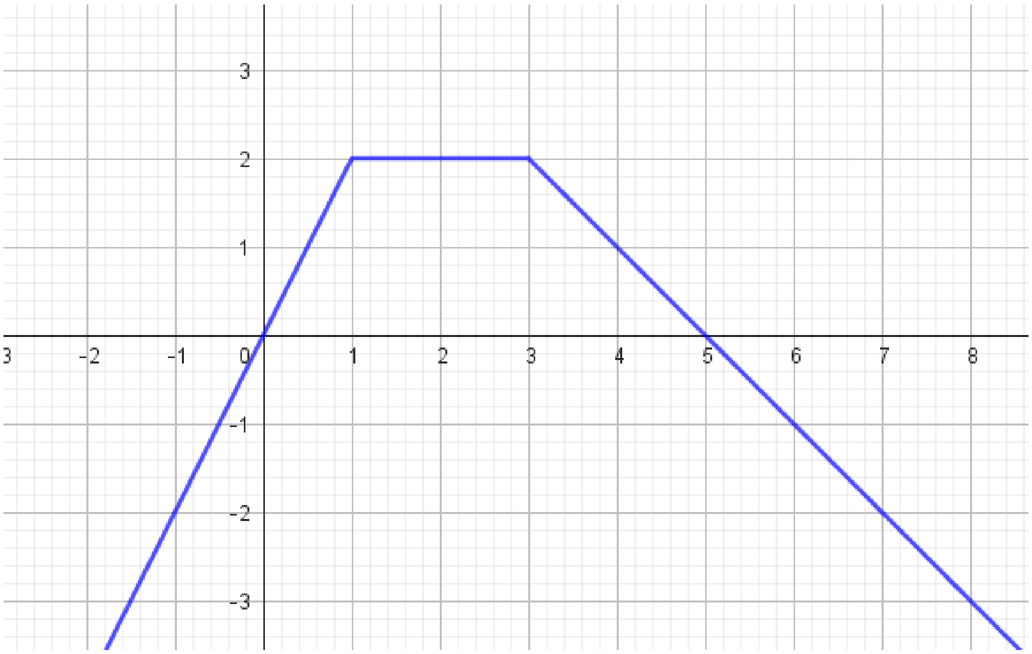


Example: Given $\int_{-1}^{1} f(x)\ dx = 5$, $\int_{1}^{4} f(x)\ dx = -12$, $\int_{4}^{5} f(x)\ dx = 4$.

The graph of $f(x)$ is given below. Find the area under the curve from $x = -1$ to $x = 5$.

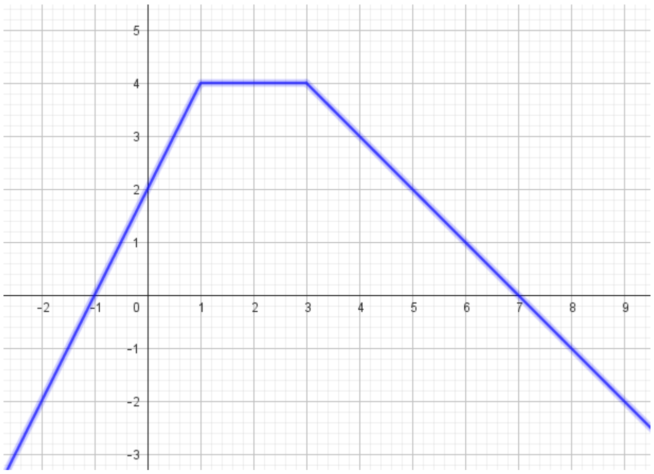


Example: The graph of $f(x)$ is given below. Compute $\int_0^6 f(x)dx = ?$



Exercise: Given $f(x)$ is an **even** and positive function, if the area under the curve $f(x)$ from $x=0$ to $x=2$ is 10, find $\int_{-2}^{2} 4f(x)dx = ?$

Exercise: The graph of $f(x)$ is given below. Compute $\int_{-2}^{3} f(x)dx = ?$



HOMEWORK: Read Section 6.1 from textbook!!! It is important to read the textbook for this section!!!