Completed

Math 2413- Calculus I

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- Check CASA calendar for due dates.
- Bring “blank notes” to class. Completed notes will be posted after class.
- Do your best to attend every lecture and lab.
- Study after every lecture; work on the quiz covering the topic we cover on the lecture immediately afterwards. Retake your quizzes for more practice.
- Get help when you need help; bring your questions to the labs, or my office hours. We also have tutoring options on campus.
- Respect your friends in class; stay away from distractive behavior. Do your best to concentrate on the lecture.
- **If you email me, mention the course code in the subject line. Email is the best way to communicate with me outside of class. Teams chat messages are not monitored or replied to**.

Chapter 6 – Integration

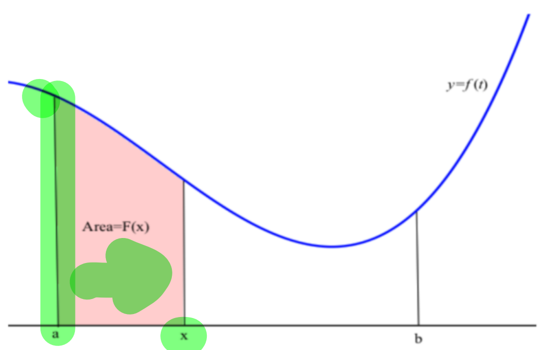
Section 6.2 - The Fundamental Theorem of Calculus

Let f be a continuous function over the interval $[a,b]$. Define a new function by

$$F(x) = \int_a^x f(t)\, dt.$$

Here, the upper limit x varies between a and b.

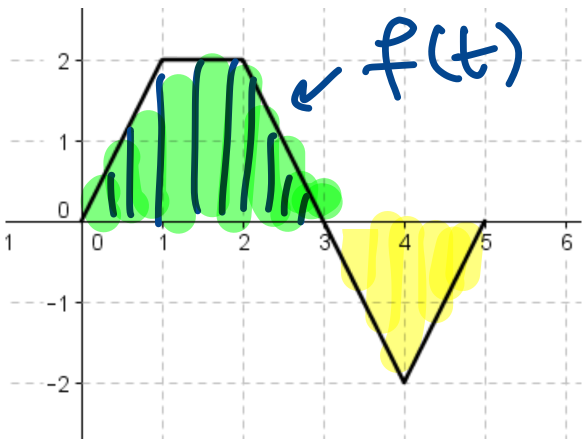
If f happens to be a nonnegative function, then $F(x)$ can be seen as the "area under the graph of f from a to x". We can think of $F(x)$ as the "accumulated area" function.



Visual: https://www.geogebra.org/m/fhmxtyw9

Example: If f is the function whose graph is given below, and $F(x)=\int_0^x f(t)\ dt$,

find the values: $F(0)$, $F(3)$ and $F(5)$.



F(x)

$F(0)=\int_0^0 f(t)\ dt=$ 0

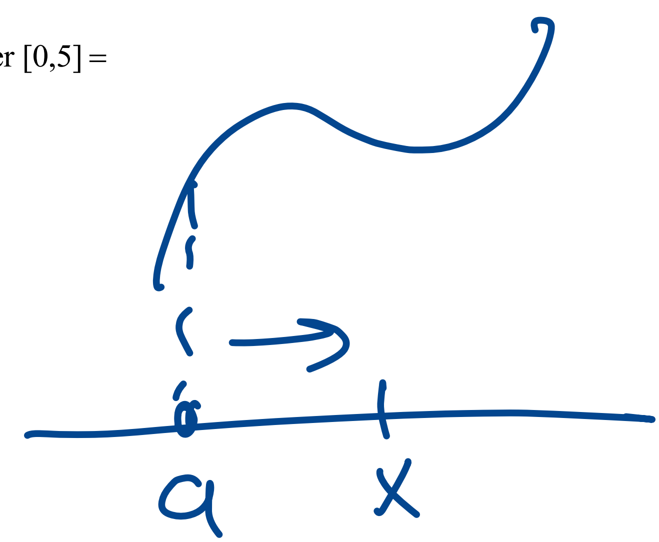
$F(3)=\int_0^3 f(t)\ dt=$ net area under the function over $[0,3]=$

$F(5)=\int_0^5 f(t)\ dt==$ net area under the function over $[0,5]=$

=

f

F



Theorem: Fundamental Theorem of Calculus Part 1

If f is a continuous function over the interval $[a,b]$, then the function

$$F(x) = \int_a^x f(t)\ dt$$

is continuous on $[a,b]$ and differentiable on (a,b). Moreover,

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\ dt = f(x), \text{ for all } x \text{ in } (a,b).$$

$$f \xrightarrow[\text{using integral}]{\text{create}} \boxed{F} \xrightarrow{\text{derivative}} f$$

Example:

If you define a function: $F(x) = \int_a^x (3t^2)\ dt$, by integrating $f(t) = 3t^2$, we know the derivative of this function: $F'(x) = 3x^2$.

$$F(x) = \int_a^x (3t^2)\ dt \xrightarrow{\text{we will see the reason later}} F(x) = x^3 + C \to F'(x) = 3x^2$$

Hence, we can use a shortcut for taking the derivative of this function:

$$F(x) = \int_a^x (3t^2)\ dt \longrightarrow \frac{d}{dx}\left[\int_a^x (3t^2)\ dt\right] = 3x^2$$

$f \xrightarrow{\int} F \xrightarrow{\text{deriv}} f$

Example: $F(x) = \int_1^x \underbrace{\left(3t^2 + 2t\right)}_{f} \, dt$, Find: $F'(x) = ?$

$=$

$F'(x) = 3x^2 + 2x$

final

Example: $\frac{d}{dx}\left[\int_4^x \frac{1}{t-2} \, dt\right] = ?$ FTOC $= \frac{1}{x-2}$

Example: $\frac{d}{dx}\left[\int_0^x 5\cos(2t) \, dt\right] = ?$ $5 \cdot \cos(2x)$

Remark: If "x" is in the lower boundary, use a property of definite integrals first.

Example: $F(x)=\int_{x}^{1}\sin(2t)\ dt$, $\quad F'(x)=?$

$$F(x)=\int_{x}^{1}\sin(2t)\ dt \xrightarrow{property} F(x)=-\int_{1}^{x}\sin(2t)\ dt \xrightarrow{derivative(shortcut)} F'(x)=-\sin(2x)$$

$\int_{1}^{x^2}$ $\qquad$ $\int_{1}^{5x^3}$

Remark: If the upper boundary is an expression involving x:

$$\frac{d}{dx}\left(\int_{a}^{u(x)} f(t)dt\right) = f(u(x))u'(x).$$

chain rule

final //

Example:

Given: $F(x)=\int_{1}^{x^2}\frac{1}{1+t}\ dt$, find the derivative: $F'(x)=?$

deriv of x^2

$$F'(x)=\frac{1}{1+x^2}\cdot(2x)=\frac{2x}{1+x^2}$$

Example:

$$\frac{d}{dx}\left(\int_{4}^{10x}\frac{1}{2t-1}\ dt\right)=?$$

FTOC

$$=\frac{1}{2(10x)-1}\cdot 10=\boxed{\frac{10}{20x-1}}$$

What if both boundaries are expressions involving x?

top bottom

FACT: $$\frac{d}{dx}\left(\int_{u(x)}^{v(x)} f(t)\,dt\right) = f\big(v(x)\big)\underline{\underline{v'(x)}} - f\big(u(x)\big)\underline{\underline{u'(x)}}$$

Example: $$F(x) = \int_{6x}^{x^2} f(t)\,dt \xrightarrow{derivative} F'(x) = f\left(x^2\right)\cdot 2x - f(6x)\cdot 6$$

Example: Given: $F(x) = \int_{x^2}^{2x^3} \left(5t^2\right) dt$, find the derivative: $F'(x) =$

$$F'(x) = 5\cdot(2x^3)^2 \cdot (6x^2) - 5(x^2)^2 \cdot (2x)$$

online quiz ✓

Exercise: Let f be a continuous function satisfying $x^3+x^2-x=\int_1^x f(t)\,dt$.

Find $f(x)$.

Find $f'(x)$.

$$x^3+x^2-x=\int_1^x f(t)\,dt$$

FTC

Take derivatives of both sides:

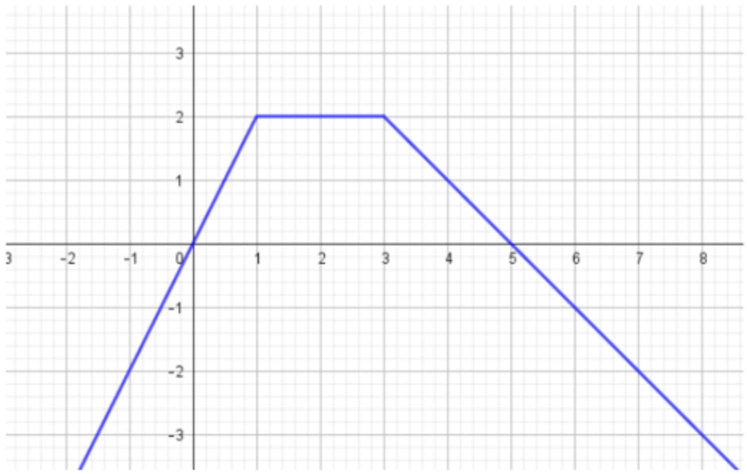
$$\boxed{3x^2+2x-1=f(x)}$$

To find $f'(x)$, take deriv.

$$f(x)=3x^2+2x-1$$

$$\Rightarrow \boxed{f'(x)=6x+2}$$

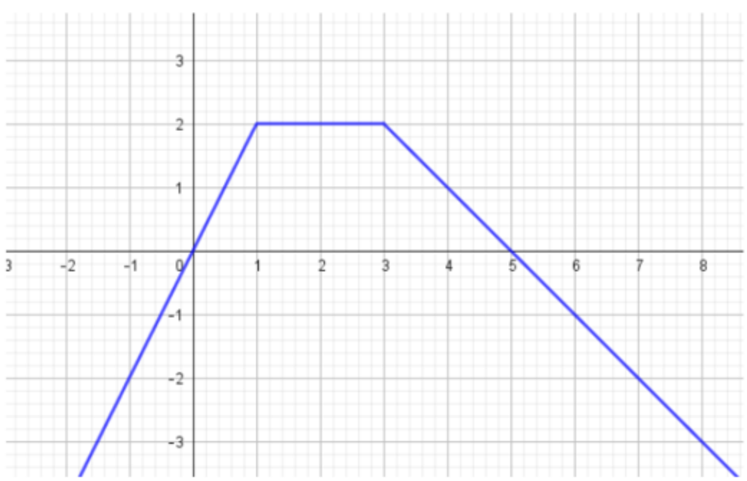
Exercise: The graph of $f(x)$ is given below. If $F(x) = \int_0^x f(t)dt$, find $F'(3) = ?$



Exercise: The graph of $f(x)$ is given below.

$$\int_0^2 f(x)dx + \int_0^3 f'(x)dx = ?$$

$$\int_3^5 \left(f(x) + f'(x) \right) dx = ?$$



HOMEWORK: READ Section 6.2 from your text book! Study the examples there.

Summary-

If we consider an integral as an "accumulation of area", then the derivative of the integral is a "rate of change" of an "accumulation of an area".

Therefore, if $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$.

- $F(a) = 0$
- Where f is positive, F is increasing
- Where f is negative, F is decreasing
- Where f is zero, F has possible max, min or inflection point
- Where f is increasing, F is concave up
- Where f is decreasing, F is concave down

Next: Fundamental Theorem of Calculus Part 2

First, what is an "anti-derivative"?

Definition: Let f be a continuous function over the interval $[a,b]$. A function G is called an **anti-derivative for** f over the interval $[a,b]$ if G is continuous on $[a,b]$ and $G'(x) = f(x)$ for all x in (a,b).

That is, an antiderivative of a function $f(x)$ is a function $G(x)$ whose derivative is $f(x)$.

Question: Why "an", not "the"?

Ex: $f(x) = 2x$; an antiderivative for this function is: $G(x) =$

Ex: $f(x) = 3x^2$; an antiderivative for this function is: $G(x) =$

Ex: $f(x) = \frac{1}{x}$; an antiderivative for this function is: $G(x) =$

Ex: $f(x) = \cos(x)$; an antiderivative for this function is: $G(x) =$

Theorem: Fundamental Theorem of Calculus Part 2

Let f be a continuous function over the interval $[a,b]$. If G is any antiderivative for f over the interval $[a,b]$, then

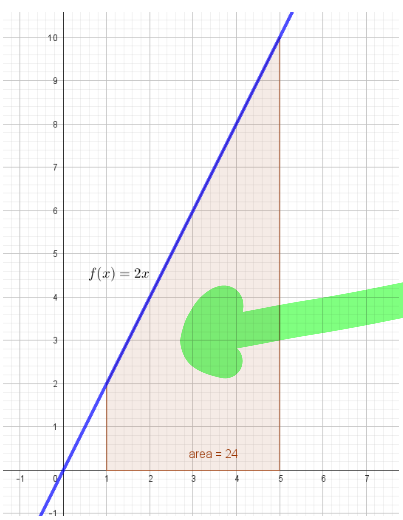
$$\int_a^b f(x)\,dx = G(b) - G(a).$$

Example: To compute: $\int_1^5 (2x)\,dx$; we first find an anti-derivative of $f(x) = 2x$;

$G(x) = x^2$

Hence: $$\int_1^5 (2x)\,dx = \left[x^2\right]_1^5 \overset{\text{plug in}}{=} (5^2) - (1^2) = 25 - 1 = 24$$

Note: Since $f(x) = 2x$ is a linear function, we can compute the area under this function on [1,5] to check that the definite integral we computed actually is equal to the area.



area = 24

Example: To compute: $\int_1^2 \left(3x^2\right)\,dx$; we first find an anti-derivative of $f(x)=3x^2$;

$G(x)=x^3$ is an antiderivative. Now, we use FTOC:

$$\int_1^2 \left(3x^2\right)\,dx = \left[x^3\right]_1^2 = \left(2^3\right)-\left(1^3\right) = 8-1=7$$

an antideriv. for $3x^2$ is $\underline{\underline{x^3}}$

Example: To compute: $\int_0^1 \left(5x^4\right)\,dx$; we first find an anti-derivative of $f(x)=5x^4$;

$G(x)=x^5$ is an antiderivative. Now, we use FTOC:

$$\int_0^1 \left(5x^4\right)\,dx = \left[x^5\right]_0^1 = \left(1^5\right)-\left(0^5\right) = 1-0=1$$

Example: Calculate the definite integral using FTOC.

$$\int_1^2 \left(5x^4 + 3x^2 + 2x\right)\,dx = \left[x^5 + x^3 + x^2\right]_1^2$$

$$= \left[(2)^5 + (2)^3 + (2)^2\right] - \left[1+1+1\right]$$

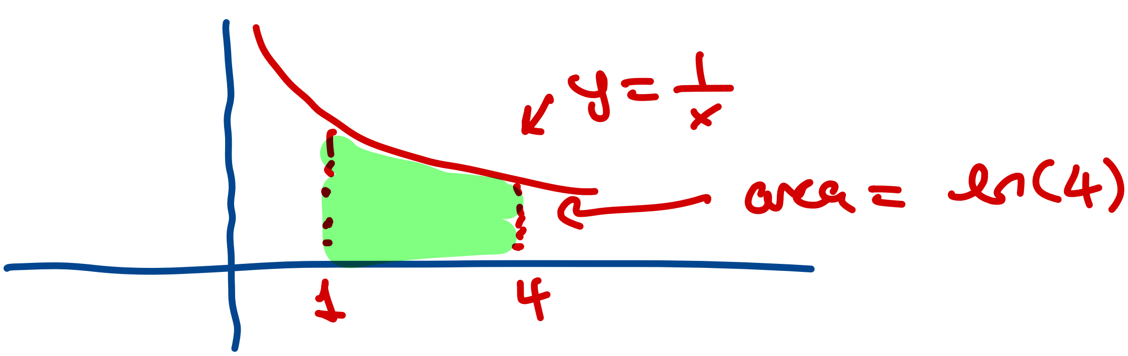
$$= (32 + 8 + 4) - (3)$$

$$= \boxed{41}$$

Example: Calculate the definite integral using FTOC.

$$\int_1^4 \frac{1}{x}\,dx = \left[\ln(x)\right]_1^4 = \ln(4) - \overset{0}{\boxed{\ln(1)}}$$

$$= \boxed{\ln(4)}$$



$\sin(x) \xrightarrow{\text{deriv.}} \cos(x) \xrightarrow{\text{integral}} \sin(x)$

integral

Example: Calculate the definite integral using FTOC.

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos(x)dx = \Big[\sin(x)\Big]_{\pi/4}^{\pi/2}$$

$$= \sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{4}\right)$$

$$= \boxed{1 - \frac{\sqrt{2}}{2}}$$

$$\int_1^2 \frac{x}{x^2+1}\,dx \qquad \int_1^2 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$$

Next: in Section 6.3, we will learn some rules to make finding the anti-derivative easier.

Need rules to compute harder integrals!

S 6.3