completed

Math 2413- Calculus I

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- Check CASA calendar for due dates.
- Bring "blank notes" to class. Completed notes will be posted after class.
- Do your best to attend every lecture and lab.
- Study after every lecture; work on the quiz covering the topic we cover on the lecture immediately afterwards. Retake your quizzes for more practice.
- Get help when you need help; bring your questions to the labs, or my office hours. We also have tutoring options on campus.
- Respect your friends in class; stay away from distractive behavior. Do your best to concentrate on the lecture.
- **If you email me, mention the course code in the subject line. Email is the best way to communicate with me outside of class. Teams chat messages are not monitored or replied to**.

Section 6.3 – Basic Integration Rules

Indefinite integral:

The notation $\int f(x)\, dx$ is used for an antiderivative of f and called an **indefinite integral.**

$$\int f(x)\, dx = F(x) \text{ means } F'(x) = f(x).$$

In general, to find $\int f(x)\, dx$, we find an antiderivative of $f(x)$, say $F(x)$, and then we write the indefinite integral as:

$$\int f(x)\, dx = F(x) + C.$$

Here, C is called the **constant of integration**.

$$\int (2x)\, dx = x^2 \boxed{+C}$$

constant

$$\int (3x^2)\, dx = x^3 + C$$

Basic Rules of Integration

1) Power Rule for Integrals

$$\int x^r \, dx = \frac{x^{r+1}}{r+1} + C, \text{ where } r \neq -1.$$

$\int x^2 dx = \frac{x^3}{3} + C$

$\int x^5 dx = \frac{x^6}{6} + C$

$\int x \, dx = \frac{x^2}{2} + C$

Reason:

$$\frac{x^{r+1}}{r+1} \xrightarrow[\text{(power rule for differentiation)}]{\textit{derivative}} \frac{(r+1)x^{r+1-1}}{r+1} = x^r$$

$$x^r \xrightarrow[\text{(power rule for integration)}]{\textit{anti-derivative}} \frac{x^{r+1}}{r+1}$$

2) $\int k \cdot f(x) \, dx = k \int f(x) \, dx$, where k is a constant number.

(Coefficients can be pulled out).

3) $\int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx$.

(If the integrand has the sum/difference of multiple functions, you can distribute the integral.)

4) $\int k \cdot \, dx = k \cdot x + C$, where k is a constant number.

(The integral of a constant number k is a line with slope k).

$\int 4 \cdot dx = 4x + C$

Using these rules, we can integrate **any polynomial**.

$\int (x^7 + 5x^4 - 4x + 2) dx$

Solved Examples:

Rule#1 - Power rule: $\int x \, dx = \frac{x^2}{2} + C$

Rule#1- Power rule: $\int x^4 \, dx = \frac{x^5}{5} + C$

Rules #1 and #2: $\int 2x^4 \, dx = 2\int x^4 \, dx = 2 \cdot \frac{x^5}{5} + C$

Rules #1 and #3: $\int \left(x + x^4\right) dx = \int x \, dx + \int x^4 \, dx = \frac{x^2}{2} + \frac{x^5}{5} + C$

Rule#4: $\int 5 \, dx = 5x + C$

Examples: Compute the following integrals:

a) $\int x^3\,dx$ $= \frac{x^4}{4} + C$

b) $\int \sqrt{x}\,dx$ $= \frac{x^{1/2+1}}{1/2+1} + C = \frac{x^{3/2}}{3/2} + C = \frac{2x^{3/2}}{3} + C$

$x^{1/2}$

c) $\int x^{-6}\,dx$ $= \frac{x^{-6+1}}{-6+1} + C = \frac{x^{-5}}{-5} + C$

power rule

d) $\int \frac{1}{\sqrt{x}}\,dx$ $= \frac{x^{-1/2+1}}{-1/2+1} + C = \frac{x^{1/2}}{1/2} + C$

$x^{-1/2}$

$= \boxed{2\sqrt{x} + C}$

Day 27

Today: 6.3 & 6.4

Thursday: 6.4 & review for final

* Schedule your final.
* Quizzes 22-26 (Ch. 6)
* Take home lab quiz due in lab on Thursday.

important questions!!!

$$\int 2x\,dx = x^2 + \underline{\underline{C}}$$

§ 6.3 Basic Rules

$$\underbrace{\int f(x)\,dx}_{} = \underbrace{F(x) + C}_{}$$

① Power Rule

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C$$

(if power $\neq -1$)

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx$$

$$F \xrightarrow{\text{deriv}} f$$

$$F \xleftarrow{\text{integrating}} f$$

*

Examples: Compute the following integrals:

a) $\int \frac{1}{x^2}\,dx = \int x^{-2} \cdot dx = \frac{x^{-2+1}}{-2+1} + c = \frac{x^{-1}}{-1} + C = \boxed{\frac{-1}{x} + C}$

power rule

b) $\int \frac{6}{x^3}\,dx$

$= \int 6 \cdot x^{-3}\,dx = 6 \cdot \underbrace{\int x^{-3}\,dx}_{\text{power rule}} = \boxed{\cancel{6} \cdot \frac{x^{-2}}{\cancel{-2}} + C} = \underbrace{\frac{-3}{x^2} + C}_{\text{final answer}}$

$\int (\text{polynomial})\,dx$

Example: Compute the integral:

$\int \left(12x^3 + 9\right)\,dx =$

$= 12 \cdot \frac{x^4}{4} + 9 \cdot x + C$

$= \boxed{3 \cdot x^4 + 9x + C} \xrightarrow{\text{derivative}} 12x^3 + 9 + 0$

Example: Compute the integral:

$$\int \left(x^8 + 5x^3 + x + 7\right) dx = \frac{x^9}{9} + 5 \cdot \frac{x^4}{4} + \frac{x^2}{2} + 7x + C$$

x^1

Know

$$\int x\,dx = \frac{x^2}{2} + C$$

power rule

$$\frac{1}{3} + 1 = \frac{4}{3}$$

Example: Compute the integral:

$$\int \left(\sqrt{x} + x^{1/3} + \frac{1}{x^4}\right) dx = \int x^{1/2} + x^{1/3} + x^{-4}\,dx$$

$$= \frac{x^{3/2}}{3/2} + \frac{x^{4/3}}{4/3} + \frac{x^{-3}}{-3} + C$$

$$= \boxed{\frac{2}{3} x^{3/2} + \frac{3}{4} \cdot x^{4/3} - \frac{1}{3} \cdot \frac{1}{x^3} + C}$$

must include "C"

$$\frac{x^1}{x^{1/2}} = x^{1 - 1/2} = x^{1/2}$$

$$\frac{x}{\sqrt{x}} = \sqrt{x} \qquad \frac{A+B}{C} = \frac{A}{C} + \frac{B}{C}$$

Example: Compute the integral:

divide

$$\int \frac{x+2\sqrt{x}}{\sqrt{x}}\,dx$$

$$= \int \left(\frac{x}{\sqrt{x}} + \frac{2\sqrt{x}}{\sqrt{x}}\right) dx = \int \left(\sqrt{x} + 2\right) dx$$

$$= \frac{x^{3/2}}{3/2} + 2 \cdot x + C$$

$$= \boxed{\frac{2}{3} \cdot x^{3/2} + 2x + C}$$

Exercise: Compute the integral:

$$\int \frac{x^5 + 5x^3 + x + 4}{x^3}\,dx$$

Divide and simplify first:

$$\int \frac{x^5 + 5x^3 + x + 4}{x^3}\,dx = \int x^2 + 5 + \frac{1}{x^2} + \frac{4}{x^3}\,dx \overset{\text{power rule on each term}}{=} \frac{x^3}{3} + 5x + \frac{x^{-1}}{-1} + 4\frac{x^{-2}}{-2} + C$$

$$\begin{tikzcd} \sin(x) \arrow[r, mapsto, "\text{deriv}"] & \boxed{\cos(x)} \arrow[l, bend left, "\text{integrate}"] \end{tikzcd}$$

$\int \cos(x)\, dx = \sin(x) + C$

Basic Formulas

Using the differentiation formulas you learned in the previous chapters, we can derive several formulas for integration.

$\cos(x) \xrightarrow{\text{deriv}} -\sin(x)$

$?? \xrightarrow{\text{deriv}} \sin(x)$

integral

$\cot(x) \xrightarrow{\text{deriv.}} -\underbrace{\csc^2(x)}$

1) Integrals of Basic Trigonometric Functions:

$$\int \sin(x)\ dx = -\cos(x) + C$$

$$\int \cos(x)\ dx = \sin(x) + C$$

$$\int \sec^2(x)\ dx = \tan(x) + C$$

$$\int \csc^2(x)\ dx = -\cot(x) + C$$

$$\int \sec(x)\tan(x)\ dx = \sec(x) + C$$

$$\int \csc(x)\cot(x)\ dx = -\csc(x) + C$$

The reasons behind these formulas are very simple:

The derivative of $\sin(x)$ is $\cos(x)$; hence, the antiderivative of $\cos(x)$ is $\sin(x)$.

$\sin(x) \xrightarrow{derivative} \cos(x)$

$\cos(x) \xrightarrow{anti-derivative \text{ (integral)}} \sin(x)$; Notation: $\int \cos(x) dx = \sin(x) + C$

Similarly:

$\tan(x) \xrightarrow{derivative} \sec^2(x)$

$\sec^2(x) \xrightarrow{anti-derivative \text{ (integral)}} \tan(x)$; Notation: $\int \sec^2(x) dx = \tan(x) + C$

$\int x^{-1}\, dx$

2) $\int \frac{1}{x}\, dx = \ln|x| + C$.

$\int \frac{1}{x}\, dx = \ln|x| + c$

must

Reason:

$\ln(x) \xrightarrow{\text{derivative}} \frac{1}{x}$

$\frac{1}{x} \xrightarrow{\text{anti-derivative (integral)}} \ln(x)$;

Notation: $\int \frac{1}{x}\, dx = \ln(|x|) + C$

(Note: To make sure ln is defined, we take the absolute value of x)

$e^x \xrightarrow{\text{deriv}} e^x$

integral

3) Integrals of Exponential Functions

$\int e^x\, dx = e^x + C$

$\int a^x\, dx = \frac{a^x}{\ln a} + C$, where $a > 0,\ a \neq 1$.

$\int 5^x\, dx = \frac{5^x}{\ln(5)} + c$

Reason:

$e^x \xrightarrow{\text{derivative}} e^x$

$e^x \xrightarrow{\text{anti-derivative (integral)}} e^x$;

$a^x \xrightarrow{\text{derivative}} a^x \ln(a)$

$a^x \ln(a) \xrightarrow{\text{anti-derivative (integral)}} a^x$

$a^x \xrightarrow{\text{anti-derivative (integral)}} \frac{a^x}{\ln(a)}$

$$\arcsin(x) \xrightarrow{\text{deriv.}} \frac{1}{\sqrt{1-x^2}}$$

4) Integrals Resulting in Inverse Trigonometric Functions

The following formulas are derived using the derivative formulas for inverse trigonometric functions (Chapter 4).

$$\int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C,$$

$$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C,$$

$$\int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \operatorname{arcsec}(x) + C.$$

Reason:

$$\arcsin(x) \xrightarrow{\textit{derivative}} \frac{1}{\sqrt{1-x^2}}$$

$$\frac{1}{\sqrt{1-x^2}} \xrightarrow{\textit{anti-derivative}\ \text{(integral)}} \arcsin(x)$$

$$\arctan(x) \xrightarrow{\textit{derivative}} \frac{1}{1+x^2}$$

$$\frac{1}{1+x^2} \xrightarrow{\textit{anti-derivative}\ \text{(integral)}} \arctan(x)$$

$$\begin{tikzcd} \sinh(x) \arrow[r, "\text{deriv.}"] & \cosh(x) \arrow[l, bend left, "\text{integral}"] \end{tikzcd}$$

5) Integrals of Hyperbolic Functions

$$\int \sinh(x)\ dx = \cosh(x) + C$$

$$\int \cosh(x)\ dx = \sinh(x) + C$$

Reason:

$$\sinh(x) \xrightarrow{\textit{derivative}} \cosh(x)$$

$$\cosh(x) \xrightarrow{\textit{anti-derivative}\ \text{(integral)}} \sinh(x)$$

TABLE OF INTEGRALS

$$\int x^r \ dx = \frac{x^{r+1}}{r+1} + C \ ; \ r \neq -1$$

$$\int \frac{1}{x} \ dx = ln|x| + C$$

$$\int sin\,x \ dx = -cos\,x + C$$

$$\int cos\,x \ dx = sin\,x + C$$

$$\int sec^2 x \ dx = tan\,x + C$$

$$\int csc^2 x \ dx = -cot\,x + C$$

$$\int sec\,x\,tan\,x \ dx = sec\,x + C$$

$$\int csc\,x\,cot\,x \ dx = -csc\,x + C$$

$$\int e^x \ dx = e^x + C$$

$$\int a^x \ dx = \frac{a^x}{ln\,a} + C \ ; \ a > 0, \ a \neq 1.$$

$$\int sinh\,x \ dx = cosh\,x + C$$

$$\int cosh\,x \ dx = sinh\,x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} \ dx = arcsin\,x + C$$

$$\int \frac{1}{1+x^2} \ dx = arctan\,x + C$$

indefinite integral: must include "+C"

Examples: Compute the integrals:

a) $\int \left(x + \frac{1}{x} \right) dx = \frac{x^2}{2} + \ln|x| + C$

b) $\int \left(5\sin(x) + 2\cos(x) \right) dx = 5 \cdot -\cos(x) + 2 \cdot \sin(x) + C$

$= \boxed{-5\cos(x) + 2\sin(x) + C}$

c) $\int \frac{4}{1+x^2}\, dx = 4 \cdot \int \frac{1}{1+x^2}\, dx = 4 \cdot \arctan(x) + C$

formula

d) $\int \frac{6}{\sqrt{1-x^2}}\, dx = 6 \cdot \int \frac{1}{\sqrt{1-x^2}}\, dx$

$= 6 \cdot \arcsin(x) + C$

warning $\int f \cdot g\, dx \neq \int f \cdot \int g$

e) $\int \left(e^x + 2^x\right)\, dx$ $= e^x + \frac{2^x}{\ln(2)} + C$

f) $\int \left(\sec(x)\tan(x) + \sec^2(x)\right)\, dx$ $= \sec(x) + \tan(x) + C$

Example: Compute: $\int \frac{x^2 - 2}{x}\, dx$

$\frac{x^2}{x}$

divide

$= \int \left(x - \frac{2}{x}\right) dx$

$\int \frac{1}{x}\, dx$

$= \frac{x^2}{2} - 2 \cdot \ln|x| + C$

$$\cancel{\int \sec(x)\cdot\tan(x)\,dx}$$

$$\cancel{= \int \sec(x)\,dx \cdot \int \tan(x)\,dx}$$

$$\cancel{\int x\cdot e^{x}\,dx \neq \int x\,dx \cdot \int e^{x}\,dx}$$

$$\int f+g \qquad \cancel{\int \frac{f}{g}}$$

Now that we know these rules, we can compute more complicated definite integrals:

$$\int_1^2 \frac{x^2-2}{x}\,dx = ?$$

$$\Big[F(x)\Big]_1^2$$

Example: Compute:

$$\int_0^\pi 4\cos(x)+2\sin(x)\,dx$$

$$= \Big[4\cdot\sin(x) - 2\cdot\cos(x) + C\Big]_0^\pi$$

optional

top

$$= \Big[4\cdot\sin(\pi) - 2\cos(\pi)\Big] - \Big[4\sin(0) - 2\cos(0)\Big]$$

$$= 0 - 2(-1) - (0 - 2\cdot 1)$$

$$= 2 - (-2) = \boxed{4}$$

Example: Compute:

$$\int_0^2 \left(2e^x - x\right)dx =$$

$$\downarrow$$

$$= \left[2\cdot e^x - \frac{x^2}{2}\right]_0^2$$

$$= \left(2e^2 - \frac{4}{2}\right) - \left(2e^0 - 0\right) = 2e^2 - 2 - 2$$

$$= \boxed{2e^2 - 4}$$

OR

$$\int (2e^x - x)\,dx = 2e^x - \frac{x^2}{2}\Bigg|_0^2$$

$$\Big[\text{antiderivative}\Big]_0^2$$

There will be times where you will be given the derivative of a function and asked to find that function:

← info

Example: Given $f'(x) = 4x + 2$; if $f(0) = 5$, find $f(x)$.

$$f(x) = \int f'(x)\,dx$$

$$f(x) = \int (4x + 2)\,dx = 2x^2 + 2x + C$$

$$f(0) = 5 \Rightarrow f(0) = 0 + 0 + c = 5$$

$$c = 5$$

$$\boxed{f(x) = 2x^2 + 2x + 5}$$

Exercise: Given $f''(x) = 6x + 12$; if $f(0) = 2$ and $f'(0) = 1$, find $f(x)$.

Attendance

Question# $$\int_1^2 \left(6x^2 - 2x + 1\right) dx = ?$$

A) 15 **B) 12** **C) 10** **D) 6** **E) None**

Question# $$\int_1^e \left(\frac{2}{x}\right) dx = ?$$

A) 2 **B) 1** **C) 0** **D) 2e** **E) None**

Question# $$\int_0^{\pi/2} \left(2\cos(x)\right) dx = ?$$

A) 2 **B) -2** **C) 1** **D) 0** **E) None**

Question# $$\int_0^{\pi/4} \left(\sec^2(x)\right) dx = ?$$

A) 2 **B) sqrt(2)** **C) 1/2** **D) 1** **E) None**

$$\int_4^6 f(x)\,dx$$

Integrating Piece-wise functions:

Example: $f(x)=\begin{cases}4x+1, & \text{if } x>1\\ 5, & \text{if } 0\le x\le 1\\ x^2+5, & \text{if } x<0\end{cases}$

Set up the integrals needed to compute: $\int_{-2}^{4} f(x)\,dx$

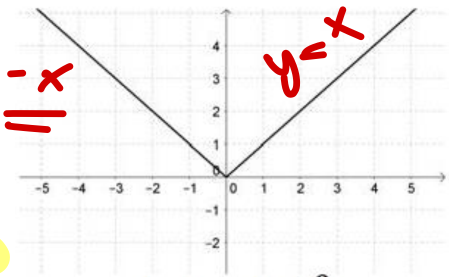
$$\int_{-2}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx$$

$$\int_{-2}^{0} (x^2+5)\,dx + \int_{0}^{1} (5)\,dx + \int_{1}^{4} (4x+1)\,dx$$

Integrals Involving Absolute Value Function

Recall that $y = |x|$ is a piecewise function!

$y = -x$ $y = x$



$$f(x) = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

$$\int_1^2 |x|\,dx = \int_1^2 x\,dx$$

$$\int_{-1}^2 |x|\,dx = \int_{-1}^0 (-x)\,dx + \int_0^2 (x)\,dx$$

To compute $\int_a^b |x|\,dx$; one might consider the integrand as a piecewise function.

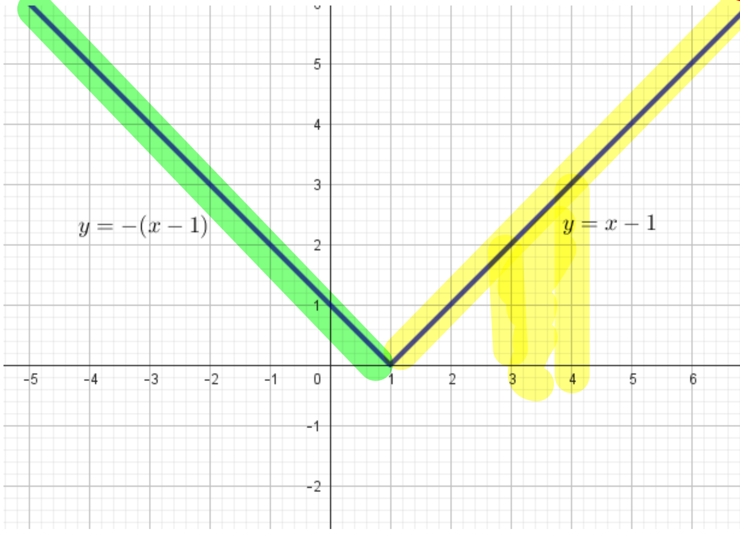
Rewrite the integral first.

$$\int_1^2 |x|\,dx \xrightarrow{[1,2];\,y=x} \int_1^2 x\,dx = \left[\frac{x^2}{2}\right]_1^2 = \frac{3}{2}$$

$$\int_{-2}^{-1} |x|\,dx \xrightarrow{[-2,-1];\,y=-x} \int_{-2}^{-1} -x\,dx = \left[-\frac{x^2}{2}\right]_{-2}^{-1} = \frac{3}{2}$$

$$\int_{-2}^{1} |x|\,dx \xrightarrow{splitup} \int_{-2}^{0} -x\,dx + \int_0^1 x\,dx = \left[-\frac{x^2}{2}\right]_{-2}^{0} + \left[\frac{x^2}{2}\right]_0^1 = 2 + \frac{1}{2} = \frac{5}{2}$$

Solved Examples: **Let's work with $f(x) = |x-1|$ over different intervals.**



The breaking point is x=1; we can rewrite this function as a piecewise function:

$$y = \begin{cases} (x-1), & \text{if } x \geq 1 \\ -(x-1), & \text{if } x < 1 \end{cases}$$

(1) Let's compute the definite integral over [2,4]:

$$\int_2^4 |x-1|\,dx \xrightarrow{\text{over the interval}[2,4]\text{; we use } y=x-1} \int_2^4 (x-1)\,dx = \left[\frac{x^2}{2} - x\right]_2^4 = (8-4)-(2-2) = 4$$

(2) Let's work with a different interval; [0,1]:

$$\int_0^1 |x-1|\,dx \xrightarrow{\text{over the interval}[0,1]\text{; we use } y=-(x-1)}$$

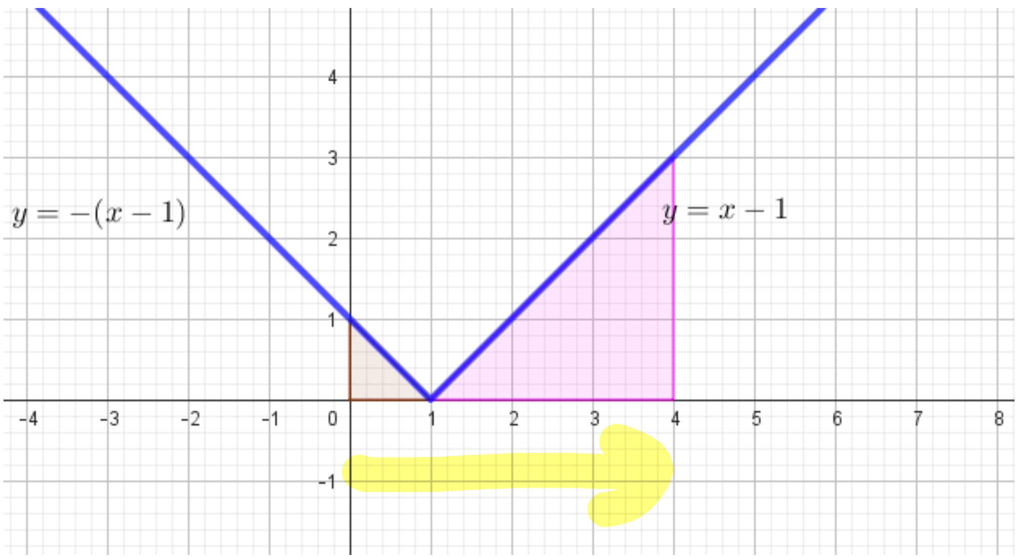
$$= \int_0^1 -(x-1)\,dx = \left[-\frac{x^2}{2} + x\right]_0^1 = (-\frac{1}{2}+1) - (0+0) = \frac{1}{2}$$

(3) Let's work over another interval: [0,4]

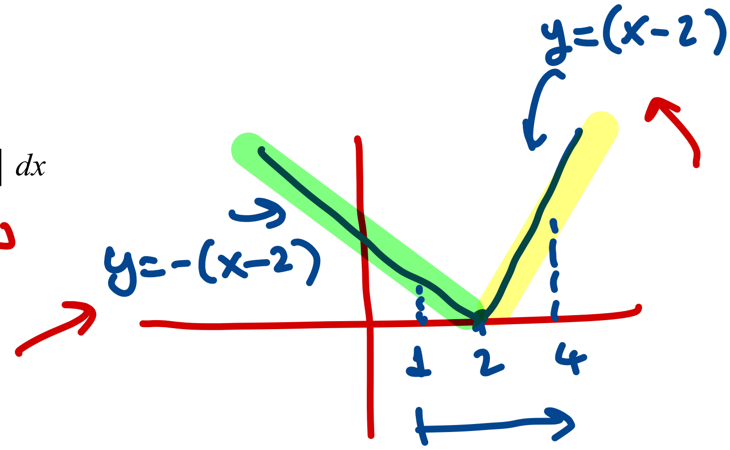
Note that this interval includes the point where the function formula changes (namely, x=1); hence, we need to split up this integral into two pieces.

$$\int_0^4 |x-1|\,dx \xrightarrow{\text{split up at x=1}} \int_0^1 |x-1|\,dx + \int_1^4 |x-1|\,dx \xrightarrow[\text{over the interval}[0,1];\text{ we use } y=-(x-1)]{\text{over the interval}[1,4];\text{ we use } y=x-1}$$

$$= \int_0^1 -(x-1)dx + \int_1^4 (x-1)dx = \left[-\frac{x^2}{2}+x\right]_0^1 + \left[\frac{x^2}{2}-x\right]_1^4 = \frac{1}{2}+\frac{9}{2} = 5$$



Example: Compute the integral: $\int_1^4 |x-2|\,dx$



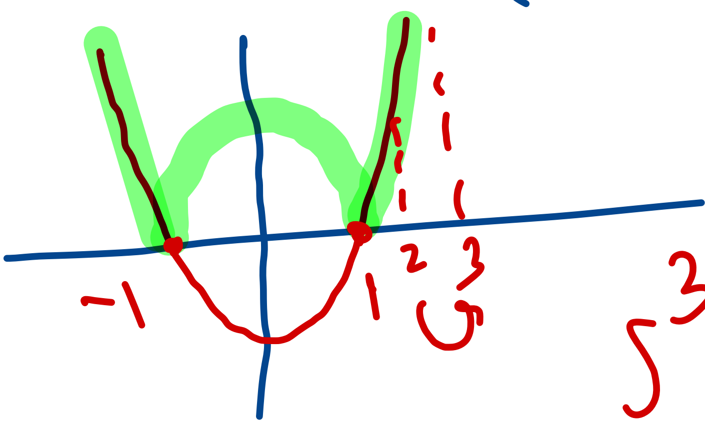
$$\int_1^4 |x-2|\,dx = \int_1^2 \underbrace{-(x-2)}_{-x+2}\,dx + \int_2^4 (x-2)\,dx$$

$$= \left[-\frac{x^2}{2}+2x\right]_1^2 + \left[\frac{x^2}{2}-2x\right]_2^4$$

$$= \dots + \dots$$

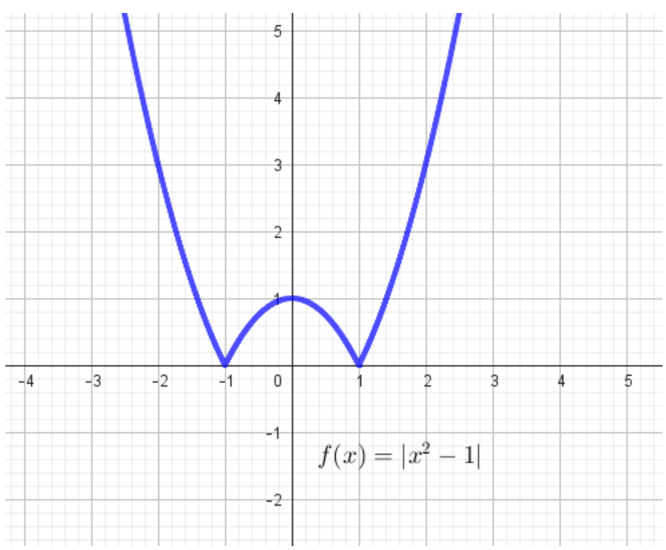
Example: Compute the integral: $\int_2^3 \left|x^2 - 1\right| \, dx$

$$|x^2-1| = \begin{cases} (x^2-1) \\ -(x^2-1) \end{cases} ?$$

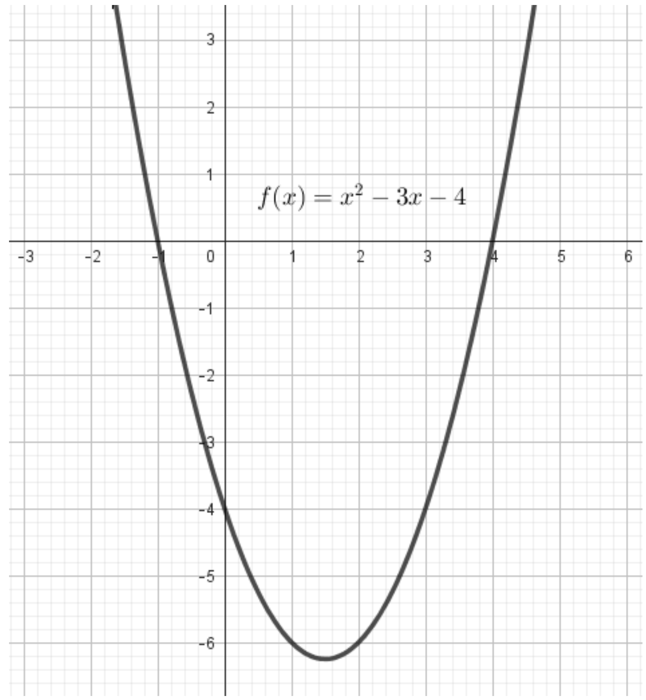


$$\int_2^3 |x^2-1|\,dx = \int_2^3 (x^2-1)\,dx$$

$$= \left[\frac{x^3}{3} - x\right]_2^3$$



Exercise: $\int_1^5 \left|x^2 - 3x - 4\right| \, dx$



EXTRA – Read S6.3 from your book.

An Application:

If the rate at which a quantity is changing is given, we can find the **net change** on the function by using integrals.

For example, if the rate at which the volume is changing is given, then the net change on the volume is found by integrating that rate.

If the rate at which an object is moving (speed) is given, we can find the total distance covered by that object by integrating the speed.

Recall: a: acceleration, v: velocity, s: position function.

$$v(t) = s'(t) \quad \text{and} \quad a(t) = v'(t) = s''(t)$$

If the acceleration is given, we can find the velocity and position functions using integration.

$$v(t) = \int_a^b a(t)\ dt$$

$$s(t) = \int_a^b v(t)\ dt$$

Total displacement vs Total Distance Covered:

$\int_{t_1}^{t_2} v(t)\ dt = s(t_1) - s(t_2)$ gives the **total displacement** of the object from time $t = t_1$ to $t = t_2$.

$\int_{t_1}^{t_2} |v(t)|\ dt =$ **total distance** traveled from time $t = t_1$ to $t = t_2$.

Very important: When we are asked to compute "total distance", we need to work with the speed function; that is, we need to use absolute value of the velocity function.

Exercise: An object moves along a line with acceleration $a(t) = 6t - 6$, where $t \geq 0$ represents the time. Initial velocity is 0 and initial position of the object is 2 units left of the origin.

Find the velocity of this object after 4 seconds.

Find the position function.

Exercise: An object moves along a line with velocity $v(t) = 6t - 12$, where $t \geq 0$ represents the time.

Find the **total displacement** of this object in the first 3 seconds.

Find the **total distance** covered by this object in the first 3 seconds.