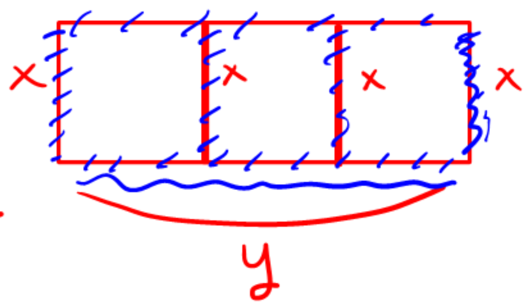
Section 5.1: Optimization

Minimum

6. How ~~much~~ fencing is required to build 3 adjacent rectangular playgrounds with equal width if the total area is to be 9,000 square feet?



$x > 0$

constraint: $A = 9{,}000$

$$x \cdot y = 9{,}000$$

$$y = 9{,}000/x$$

Perimeter: $4x + 2y$ to be minimized

$$P = 4x + 2y = 4x + 2 \cdot \frac{9{,}000}{x}$$

$$P = 4x + \frac{18{,}000}{x}$$

$$\boxed{P'} = 4 - \frac{18{,}000}{x^2} = 0$$

$$\Rightarrow \quad 4 = \frac{18{,}000}{x^2} \quad \Rightarrow \quad x^2 = \frac{18{,}000}{4}$$

$$x^2 = 4500$$

$$x = \cancel{\pm}\sqrt{4500}$$

Critical point: $x = \sqrt{4500} = 30\sqrt{5}$

Check: $P''(x) = 0 - 18{,}000 \cdot \frac{-2}{x^3} = \frac{36{,}000}{x^3}$

$$p''(30\sqrt{5}) = \frac{36{,}000}{(30\sqrt{5})^3} > 0$$

$\cup$

c.p. is a min ✓

to minimize fencing: $x = 30\sqrt{5}$

$$y = \frac{9000}{x}$$

$$y = \frac{\cancel{9000}^{300}}{\cancel{30}\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}$$

$$y = \frac{\cancel{300}\sqrt{5}}{\cancel{5}}$$

$$y = 60\sqrt{5}$$

$x: 30\sqrt{5}$ $y: 60\sqrt{5}$

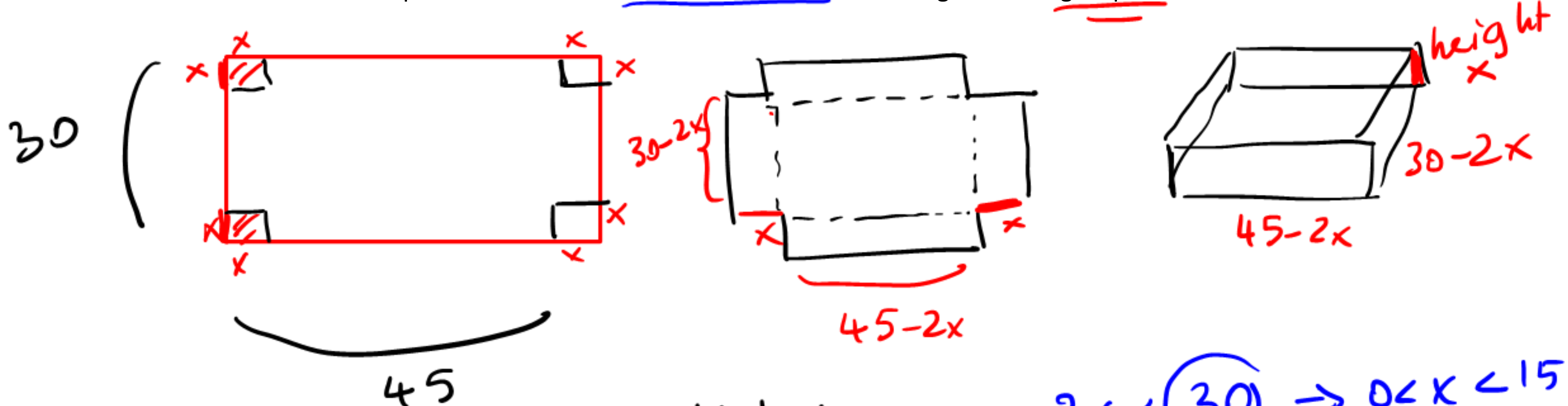
min. perimeter: $4x + 2y$

$$4 \cdot 30\sqrt{5} + 2 \cdot 60\sqrt{5}$$

$$= \boxed{240\sqrt{5}}$$

goal: max volume

14. An open top box with a rectangular base is to be made from a rectangular piece of cardboard that measures 30 cm by 45 cm. A square will be cut from each corner of the cardboard and the sides will be turned up to form the box. Find the dimensions that will give the largest possible volume.



Note: $x > 0$, $2x < 30 \Rightarrow 0 < x < 15$

shorter side

$$V = x \cdot (30-2x) \cdot (45-2x)$$

to be maximized.

$V =$ expand...

$$V' = 12x^2 - 300x + 1350 = 0$$

quadratic formula: $x_1 = 5.9$, $\cancel{x_2 = 19.1}$

one critical point

not in domain

$$V'' = 24x - 300$$

$$V''(5.9) = 24(5.9) - 300 < 0$$

max ✓

To maximize volume, $x = 5.9$

Dimensions:

x by $45 - 2x$ by $30 - 2x$

$45 - 2(5.9)$
11.8

5.9 by $45 - 11.8$ by $30 - 11.8$

★ $\boxed{5.9 \text{ cm by } 33.2 \text{ cm by } 18.2 \text{ cm}}$

largest volume?

$V = 5.9 \cdot 33.2 \cdot 18.2$

$20 \; m^3$

17. A cylindrical can that has a capacity of 20 m3 will be made. The metal used to build the top costs $10 per square meter while the bottom costs $20 per square meter. The material used for the side costs $15 per square meter. What are the dimensions that will minimize the cost?

$V = 20 \; m^3$

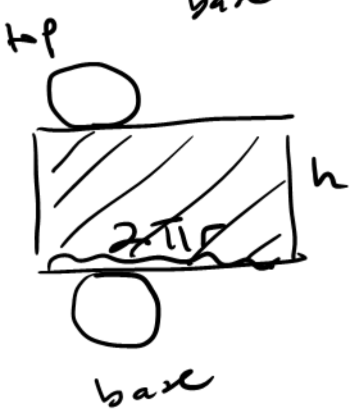
$$\text{area} = \pi r^2 + \pi r^2 + 2\pi r \cdot h$$

base top side

goal: to minimize cost

$$C = 10 \cdot \pi r^2 + 20\pi r^2 + 15 \cdot 2\pi r h$$

to be min.



$$C = 30\pi r^2 + 30 \cdot \pi r h$$

?

$$V = 20 = \pi r^2 \cdot \boxed{h} \Rightarrow h = \frac{20}{\pi r^2}$$

$$C = 30\pi r^2 + 30 \cancel{\pi} \cancel{r} \cdot \frac{20}{\cancel{\pi} r^{\cancel{2}}}$$

$$C = 30\pi r^2 + \frac{600}{r}$$

to be min.

$$C' = 60\pi r - \frac{600}{r^2} = 0$$

$$60\pi r = \frac{600}{r^2}$$ by r^2

$$\frac{60\pi r^3}{60\pi} = \frac{\cancel{6}00}{\cancel{6}0\pi}$$

$$r^3 = \frac{10}{\pi}$$

$$r = \sqrt[3]{\frac{10}{\pi}}$$ critical point.

$$C'' = 60\pi + \frac{1200}{r^3}$$

$$C''\left(\sqrt[3]{\frac{10}{\pi}}\right) = 60\pi + \frac{1200}{\frac{10}{\pi}} > 0$$

min at $r = \sqrt[3]{\frac{10}{\pi}}$ ✓

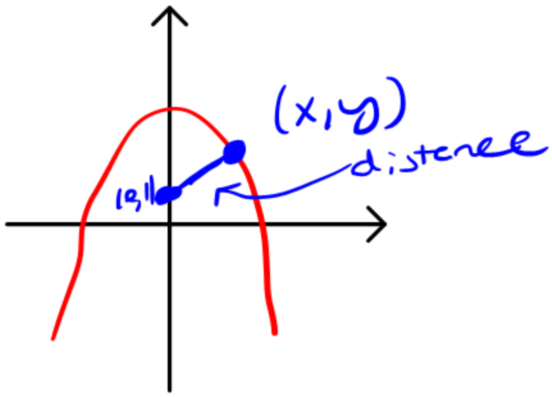
dimensions

$$r = \sqrt[3]{\frac{10}{\pi}} \qquad h = \frac{20}{\pi \cdot \sqrt[3]{\frac{100}{\pi^2}}}$$ //

$$h = \frac{20}{\pi r^2}$$

(x,y) $(0,1)$

18. Find the coordinates of the point(s) on the parabola y = 4 - x^2 that is closest to the point (0,1).



$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

$$d = \sqrt{x^2 + (y-1)^2}$$

$y = 4-x^2$

$$d = \sqrt{x^2 + (4-x^2-1)^2}$$

$$d = \sqrt{x^2 + (3-x^2)^2}$$ to be min.

critical points? $d' = 0$?

easier new function $D = x^2 + (3-x^2)^2$

if D min then d min. $\sqrt{D}$

$$D = x^2 + (3-x^2)^2$$ to be min.

$$D' = 2x + 2(3-x^2)\cdot(-2x)$$ chain

$$D' = 2x - 12x + 4x^3$$

$$D' = -10x + 4x^3 = 0$$

$$2x(-5 + 2x^2) = 0$$

$$\Rightarrow \underline{x = 0} \quad \text{or} \quad -5 + 2x^2 = 0$$

$$2x^2 = 5$$

$$x^2 = \frac{5}{2}$$

(x, y) is a point

↑ can be 0, or negative

$$x = \pm\sqrt{\frac{5}{2}}$$

3 critical points: $0, +\sqrt{\frac{5}{2}}, -\sqrt{\frac{5}{2}}$

Check

$$D'' = -10 + 12x^2$$

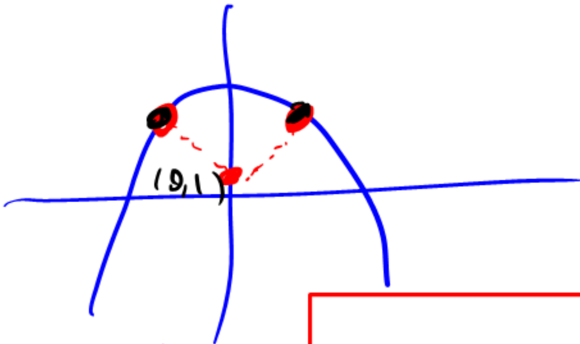
$D''(0) = -10 < 0$ ∩ max ✗

$D''\left(\sqrt{\frac{5}{2}}\right) = -10 + 12 \cdot \frac{5}{2} > 0$ ∪ min ✓

$D''\left(-\sqrt{\frac{5}{2}}\right) = -10 + 12 \cdot \frac{5}{2} > 0$ ∪ min ✓

to minimize D

$$x = \sqrt{\frac{5}{2}} \quad \text{or} \quad x = -\sqrt{\frac{5}{2}}$$



$$y = 4 - \underline{\underline{x^2}} = 4 - \frac{5}{2} = \frac{3}{2}$$

Closest Points

$$\boxed{\left(\sqrt{\frac{5}{2}}, \frac{3}{2}\right) \ \& \ \left(-\sqrt{\frac{5}{2}}, \frac{3}{2}\right)}$$

warning if minimum distance?

$$d = \sqrt{\ \cdots\ }$$

$$\left(\sqrt{\frac{5}{2}}, \frac{3}{2}\right) \ \& \ (0, 1)$$

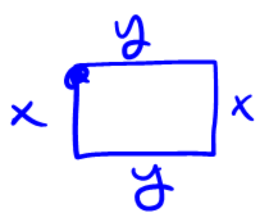
$$d = \sqrt{\frac{5}{2} + \frac{1}{4}} = \sqrt{\frac{11}{4}}$$

minimum distance : $d = \frac{\sqrt{11}}{2}$

max area?

22. A string of length 20 inches is to be fold to make a rectangle. What are the dimensions of the rectangle with largest possible area?

20 in



$P = 20$

$2x + 2y = 20$

$x + y = 10 \Rightarrow y = 10 - x$

$A = x \cdot y = x \cdot (10 - x)$ to be max.

$A = 10x - x^2$

$A' = 10 - 2x = 0$

$\Rightarrow 2x = 10 \Rightarrow x = 5$ c.p.

check $A'' = -2$ $\quad A''(5) = -2 < 0$

max at $x = 5$ ✓

dimensions: x by y

5 by $10 - 5 = 5$

$\boxed{5 \text{ by } 5}$

largest area: $5 \cdot 5 = \boxed{25}$

$x = ?$

30. A manufacturer estimates that the cost of producing x items can be modeled by C(x)=0.001x^2 - 0.6x + 1,000. How many items should be produced to minimize the cost?

$$C(x) = 0.001x^2 - 0.6x + 1{,}000$$

to be min..

$$C'(x) = 0.002x - 0.6 = 0$$

$$\Rightarrow \quad 0.002x = 0.6$$

$$x = \frac{0.6}{0.002} = \frac{600}{2} = 300$$

$\times$ by 1000

c.p: $x = 300$

check $C'' = 0.002 > 0$

min at 300

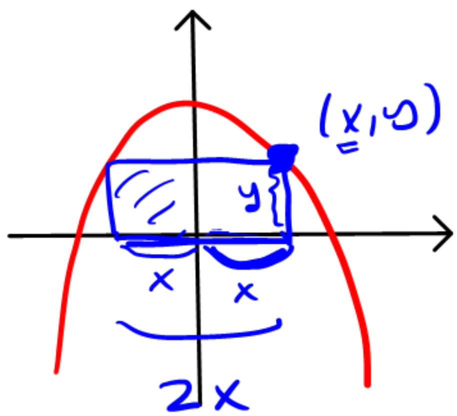
$$\boxed{x = 300}$$

if also min cost. (min. value)

$C(300) = \ldots$

1) Find the largest possible area for a rectangle with base on the x-axis and upper vertices on the curve y=12-x2.

$$y = 12 - x^2$$



$$A = 2x \cdot \underline{y}$$ max

$$A = 2x(12 - x^2)$$

$$A = 24x - 2x^3$$

$$A' = 24 - 6x^2 = 0$$

$$6x^2 = 24$$

$$x^2 = 4$$

$$x = \pm 2$$

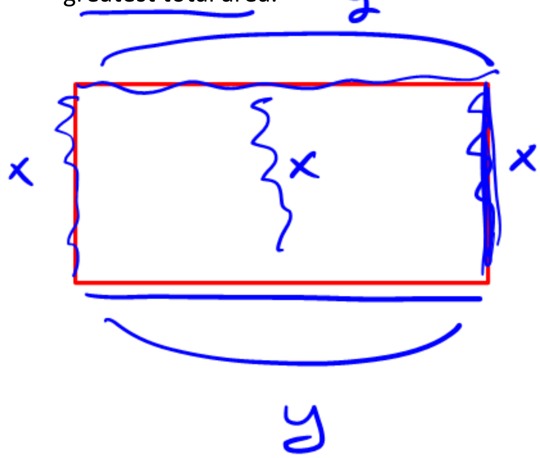
c.p. $\boxed{x = 2}$

$$A'' = -12x \qquad A''(2) = -24 < 0$$

@ max

$$A = 2x\underbrace{y}_{12-x^2} = 2 \cdot 2 \cdot (12 - 4) = 4 \cdot 8 = \boxed{32}$$

2) A rectangular playground is to be fenced off and divided into two parts by a fence parallel to one side of the playground. 1200 feet of fencing is used. Find the dimensions of the playground that will enclose the greatest total area.



$$P = 1200$$

$$3x + 2y = 1200$$

$$y = \frac{1200 - 3x}{2}$$

$$A = x \cdot y \quad \text{to be maximized.}$$

$$A = x \cdot \left(\frac{1200 - 3x}{2} \right)$$

$$A = x \left(600 - \frac{3}{2} x \right)$$

$$A = 600x - \frac{3}{2} x^2$$

$$A' = 600 - 3x = 0$$

$$3x = 600$$

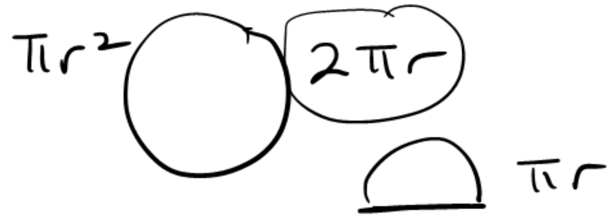
c.p. $\boxed{x = 200}$

fill in details

Check!

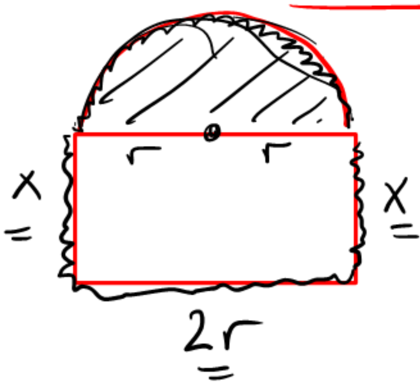
$$y = \frac{1200 - 3 \cdot 200}{2} = 300$$

guess shape?



3) The figure below shows a region that consists of a semi-circle on top of a rectangle. Give the value of x that maximizes the area of the region if the circumference of the region is 8.

$A = \frac{\pi r^2}{2}$



given $8 = \pi r + 2r + 2x$

$2x = 8 - \pi r - 2r$

$x = \frac{8 - \pi r - 2r}{2}$

$A = 2r x + \frac{\pi r^2}{2}$ to be max

$A = \ldots$ rewriting in terms of r

$A' = 0$ solve & find critical points.

$\underline{a} \longmapsto a+h$

$27 \xrightarrow{h=1} 28$

approx

Section 5.2: Differentials

$df = f'(a) \cdot h$

8. (28)^2/3

$28^{2/3}$

$f(x) = \boxed{x^{2/3}} \quad a = 27 \quad h = 1$

$df = f'(x) \cdot h$

$f(27) = 9$

$df = \frac{2}{3} \cdot (27)^{-1/3} \cdot (1)$

differential

$f(28) = ?$

$df = \frac{2}{3} \cdot \frac{1}{3} \cdot 1 = \frac{2}{9}$

14. sin(58^0)

$28^{2/3} \approx \boxed{f(27)} + df$

$(27)^{2/3} + \frac{2}{9}$

$f(28) \approx 9 + \frac{2}{9} = \boxed{\frac{82}{9}}$

$\sin(58^\circ)$

$60^\circ \to \frac{\pi}{3}$

$-2^\circ \to -\frac{\pi}{180} \cdot 2 = -\frac{\pi}{90}$

★ radians!

$60 \xrightarrow{h=-2} 58$

$f(x) = \sin x \quad a = \frac{\pi}{3} \quad h = -\frac{\pi}{90}$

$$df = f'(x) \cdot h$$

$$df = \cos x \cdot h$$

$$df = \cos\left(\frac{\pi}{3}\right) \cdot \frac{-\pi}{90}$$

$$df = \frac{1}{2} \cdot \frac{-\pi}{90} = \frac{-\pi}{180}$$

$$\sin(58^\circ) \approx \sin(60^\circ) + df$$

$$\approx \boxed{\frac{\sqrt{3}}{2} + \frac{-\pi}{180}}$$

$$\frac{\sqrt{3}}{2} - \frac{\pi}{180} \quad //$$

known

$f(x)$ $\quad x \longmapsto x+h$

$$f(x+h) = ? \approx f(x) + \underline{\underline{df}}$$

differential

find differential

18. f(x)=x^3+x, a=3, h=0.5

$a = 3, \quad h = 0.5$

$f(x) = x^3 + x \Rightarrow f'(x) = 3x^2 + 1$

$df = f'(a) \cdot h$

$df = f'(3) \cdot (0.5) = 28 \cdot (0.5) = \boxed{14}$

$3 \longmapsto 3.2$

$h = 0.2$

25. Given f?(x)=(x^3+5)^1/5 and f(3)=2, use differentials to estimate f(3.2).

$f'(x) = (x^3+5)^{1/5}$ $\qquad f(3) = 2$

$f(3.2) = ?$ approx ?

$f(3.2) \approx f(3) + df$???

$df = f'(a) \cdot h = f'(3) \cdot (0.2)$

known value

$df = (3^3+5)^{1/5} \cdot \frac{1}{5}$

$df = 32^{1/5} \cdot \frac{1}{5} = 2 \cdot \frac{1}{5} = \frac{2}{5}$

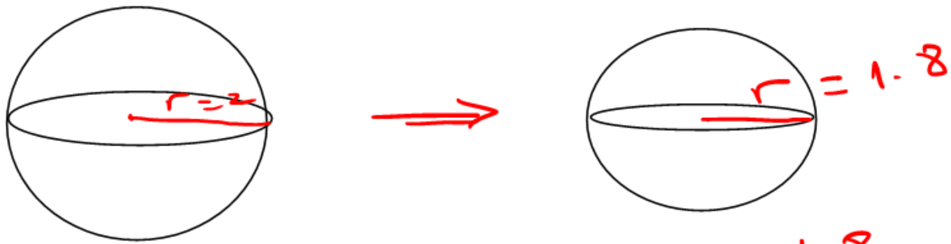
$$f(3.2) \approx \underbrace{f(3)}_{\text{given}} + \underbrace{df}_{\text{found}}$$

$$2 + \frac{2}{5}$$

$$f(3.2) \approx \boxed{\frac{12}{5}}$$ our approximation

differential

32. Use differentials to approximate the change in the volume of a spherical balloon of radius 2 meters if the balloon deflates to a radius of 1.8 meters.



$r = 2$ $\qquad$ $r = 1.8$

Volume changed.

How much did the volume change?

exact change : $V_{initial} - V_{final}$

$V = \frac{4}{3}\pi r^3$ $\qquad$ $\frac{4}{3}\pi(2)^3 - \frac{4}{3}\pi(1.8)^3$

$= \ldots$ calculate.

approximate the change :

$$V_{final} \approx V_{initial} + df$$

$$2 \xrightarrow{h} 1.8$$

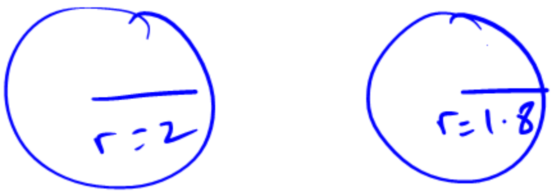
$$V = \frac{4}{3}\pi r^3 \qquad r = 2 \qquad h = -0.2$$

$$dV = (4\pi r^2) \cdot h = 4\pi \cdot 2^2 \cdot (-0.2)$$

$$dV = \boxed{-3.2\pi}$$

change in volume

$$\Delta V \approx df = -3.2\pi$$



volume will decrease by 3.2π

change $\approx$ df

5.3

$\frac{0}{0}$ or $\frac{\infty}{\infty}$ $\Rightarrow$ L'Hosp.

L'Hospital's Rule

$$\lim \frac{f}{g} = \lim \boxed{\frac{f'}{g'}} = \cdots$$

46.) lim x->infinity (lnx/cotx)

$$\lim_{x\to\infty} \frac{\ln x}{\cot x} \to \frac{\infty}{\text{oscill}} \quad \boxed{DNE}$$

if

$$\lim_{\boxed{x\to 0^+}} \frac{\ln x}{\cot x} \to \frac{-\infty}{+\infty} \quad \boxed{\frac{\infty}{\infty}}$$

Indeterminate

state!

$$\lim_{x\to 0^+} \frac{\ln x}{\cot x} \underset{\frac{\infty}{\infty}}{\overset{\text{L'Hosp.}}{=}} \lim_{x\to 0^+} \frac{1/x}{-\csc^2 x}$$

$$= \lim_{x\to 0^+} -\frac{\sin^2 x}{x} = \frac{0}{0}$$

$$\underset{\frac{0}{0}}{\overset{\text{L'Hosp.}}{=}} \lim_{x\to 0^+} -\frac{2\sin x \cdot \cos x}{1}$$

$$\underset{\text{plug in 0}}{=} -2\cdot 0\cdot 1 = \boxed{0}$$

$\cos(0) \to 1^{\infty}$ indet. ✓ "Ln trick"

62.) lim x->infinity (cos 1/x)^x

$$\lim_{x\to\infty}\left[\cos\left(\frac{1}{x}\right)\right]^{x} = \boxed{1}$$

final answer

rewrite

$$= \lim_{x\to\infty} e^{\ln\left(\cos\left(\frac{1}{x}\right)\right)^{x}}$$

$$= \lim_{x\to\infty} e^{x\cdot\ln\left(\cos\left(\frac{1}{x}\right)\right)}$$

$$= e^{\boxed{\lim_{x\to\infty} x\ln\left(\cos\left(\frac{1}{x}\right)\right)}} = e^{\#} = e^{0}$$

next page

final $\boxed{= 1}$

$\infty \cdot \ln 1 = \infty \cdot 0$

work: $\lim_{x\to\infty} x\cdot\ln\left(\cos\left(\frac{1}{x}\right)\right)$

rewrite

$$\lim_{x\to\infty}\frac{\ln\left(\cos\left(1/x\right)\right)}{1/x} \qquad \frac{0}{0}$$

$\ln(u) \to \frac{u'}{u}$

$u = \cos\left(\frac{1}{x}\right)$

$$\underset{\frac{0}{0}}{\overset{\text{L'Hop}}{=}} \lim_{x \to \infty} \frac{\dfrac{-\sin\left(\frac{1}{x}\right) \cdot \cancel{\boxed{\frac{-1}{x^2}}}}{\cos\left(\frac{1}{x}\right)}}{\cancel{\boxed{-\frac{1}{x^2}}}}$$

$$\overset{\text{rewrite}}{=} \lim_{x \to \infty} -\frac{\sin\left(1/x\right)}{\cos\left(1/x\right)}$$

$$\frac{1}{\infty} \to 0$$

$$\overset{\text{plug in "}\infty\text{"}}{=} -\frac{\sin(0)}{\cos(0)} = -\frac{0}{1} = \underline{\underline{0}}$$

go back!

famous

get used to it!

68.) lim x->infinity (1 - 2/x)^3x

$(1-0)^{\infty} = 1^{\infty}$ Indeter.

"Ln" trick

idea

$$\boxed{e^{\ln p} = p}$$

$$\lim_{x\to\infty}\left(1-\frac{2}{x}\right)^{3x}$$

$$= \lim_{x\to\infty} e^{\ln\left(1-\frac{2}{x}\right)^{3x}}$$

$$= \lim_{x\to\infty} e^{3x\cdot\ln\left(1-\frac{2}{x}\right)}$$

$$= e^{\boxed{\lim_{x\to\infty} 3x\cdot\ln\left(1-\frac{2}{x}\right)}}$$??

find it on next page

$$= e^{\#??}$$

$$= \boxed{e^{-6}}$$

final answer

$$\lim_{x\to\infty} 3x \cdot \ln\left(1-\frac{2}{x}\right)$$

$\infty \cdot \ln 1 = \infty \cdot 0$

Indet.

rewrite as $\frac{\infty}{\infty}$ or $\frac{0}{0}$

rewrite

$$= \lim_{x\to\infty} \frac{3\ln\left(1-\frac{2}{x}\right)}{1/x} \qquad \frac{0}{0}$$

L'Hosp.

$$\underset{\frac{0}{0}}{=} \lim_{x\to\infty} \frac{3 \cdot \frac{+\frac{2}{x^2}}{1-\frac{2}{x}}}{-\frac{1}{x^2}}$$

rewrite

cancel

rewrite

$$= \lim_{x\to\infty} 3 \cdot \frac{\boxed{2/x^2}^{\,2}}{1-\frac{2}{x}} \cdot \frac{-x^2}{1}$$

$$= \lim_{x\to\infty} \frac{-6}{1-\frac{2}{x}}$$

plug in

$$= \frac{-6}{1-0}$$

$$= \frac{-6}{1} = -6$$

⇑

go back!

76.) limx-> infinity ((sqrt x) + 1)^(1/(sqrt x))

$$\lim_{x\to\infty} \left(\sqrt{x}+1\right)^{\frac{1}{\sqrt{x}}}$$

$x\to\infty$

$\sqrt{x}\to\infty$

$\boxed{\infty^0}$ indet. form

"ln" trick

$$=\lim_{x\to\infty} e^{\ln(\sqrt{x}+1)^{1/\sqrt{x}}}$$

$$=\lim_{x\to\infty} e^{\frac{1}{\sqrt{x}}\ln(\sqrt{x}+1)}$$

$$= e^{\boxed{\lim_{x\to\infty}\frac{1}{\sqrt{x}}\ln(\sqrt{x}+1)}}$$

next page

$$= e^{\boxed{0}}$$

← see next page

$$= \boxed{1}$$

//

$$\lim_{x\to\infty} \frac{1}{\sqrt{x}} \cdot \ln(\sqrt{x}+1)$$

$0 \cdot \infty$

L'Hosp? X

rewrite

as $\frac{0}{0}$ or $\frac{\infty}{\infty}$

$$= \lim_{x\to\infty} \frac{\ln(\sqrt{x}+1)}{\sqrt{x}}$$

$\boxed{\frac{\infty}{\infty}}$

indet.

L'Hosp.

$$\underset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{\frac{\cancel{\frac{1}{2\sqrt{x}}}}{\sqrt{x}+1}}{\cancel{\frac{1}{2\sqrt{x}}}}$$

cancel

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x}+1} \to \frac{1}{\infty} \to 0$$

$$= 0$$

go back!

Ex: Calculate the limit: limx?0((6/x)?6cot(x)). ??

$$\boxed{\lim_{x\to 0}\left(\frac{6}{x}\right)\cdot 6\cot(x)}$$

or

$$\lim_{x\to 0^+}\left(\frac{6}{x}\right)^{6\cot x}$$

say $x\to 0^+$ $\downarrow$ $\infty\cdot\infty$ or $-\infty$

∞

if

$$\lim_{x\to 0^+}\left(\frac{6}{x}\right)^{6\cot x}$$

∞^{∞} ?? indet.

"Ln" trick

$$= \lim_{x\to 0^+} e^{\ln\left(\frac{6}{x}\right)^{6\cot x}}$$

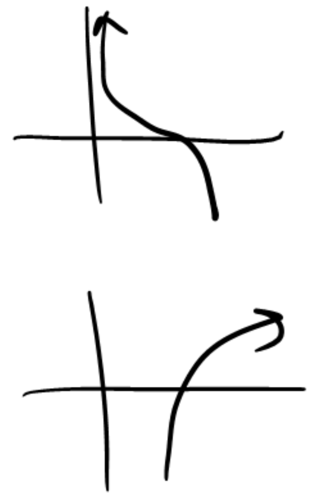
$$= \lim_{x\to 0^+} e^{6\cot x\cdot\ln\left(\frac{6}{x}\right)}$$

$$= e^{\boxed{\lim_{x\to 0} 6\cot x\cdot\ln\left(\frac{6}{x}\right)}} \to e^{\infty} \to \boxed{\infty}$$

$$\lim_{x \to 0^+} 6\cot x \cdot \ln\left(\underset{\to\infty}{\frac{6}{x}}\right)$$

$$\infty \cdot \infty$$

$$\infty$$



problems
(5,6,7,9,10) from this Thursday's emcf (11/20).

posted the hws

check the key