Motivational Problems
3 -Integration and its Applications

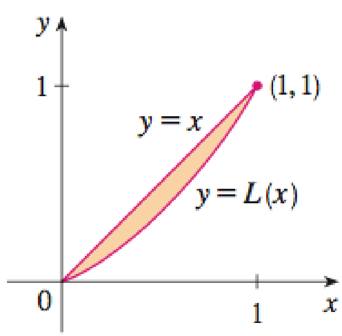
MAT1450: Accelerated Calculus   Sec 17957   Fall 2017

Due date is in class on Oct 26, 2017 - optional

1. **Income equality?** Economists use a cumulative distribution called a *Lorenz curve* to describe the distribution of income between households in a given country. Typically, a Lorenz curve is defined on [0,1] with endpoints (0, 0) and (1, 1), and is continuous, increasing, and concave upward. The points on this curve are determined by ranking all households by income and then computing the percentage of households whose income is less than or equal to a given percentage of the total income of the country.

   For example, the point (a/100, b/100) is on the Lorenz curve if the bottom a% of the households receive less than or equal to b% of the total income. Absolute equality of income distribution would occur if the bottom a% of the households receive a% of the income, in which case the Lorenz curve would be the line $y = x$.

   The area between the Lorenz curve and the line $y = x$ measures how much the income distribution differs from absolute equality. The coefficient of inequality is the ratio of the area between the Lorenz curve and the line $y = x$ to the area under $y = x$.

   (a) Show that the coefficient of inequality is twice the area between the Lorenz curve and the line $y = x$, that is, show that

   $$\text{coefficient of inequality } = 2\int_0^1 [x - L(x)]dx$$

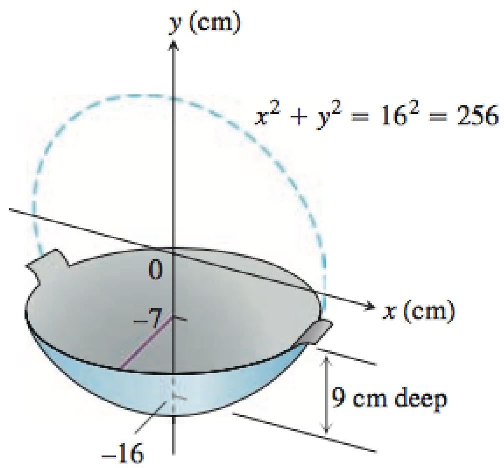
(b) The income distribution for a certain country is represented by the Lorenz curve defined by the equation

$$L(x) = \frac{5}{12}x^2 + \frac{7}{12}x.$$

What is the percentage of total income received by the bottom 50% of the households? Find the coefficient of inequality.

2. **Mathematics of breathing.** Breathing is cyclic and a full respiratory cycle from the beginning of inhalation to the end of exhalation takes about 5 s. The maximum rate of air flow into the lungs is about 0.5 L/s. This explains, in part, why the function $f(t) = \frac{1}{2}\sin(2\pi t/5)$ has often been used to model the rate of air flow into the lungs. Use this model to find the volume of inhaled air in the lungs at time t.

3. **Designing a wok.** You are designing a wok frying pan that will be shaped like a spherical bowl with handles. A bit of experimentation at home persuades you that you can get one that holds about 3 L if you make it 9 cm deep and give the sphere a radius of 16 cm. To be sure, you picture the wok as a solid of revolution, as shown here, and calculate its volume with an integral.



To the nearest cubic centimeter, what volume do you really get?

4. **Skydiving.** Skydivers Alice and Bob are in a helicopter hovering at 6400 ft. Alice jumps and descends for 4 sec before opening her parachute. The helicopter then climbs to 7000 ft and hovers there. Forty-five seconds after Alice leaves the aircraft, Bob jumps and descends for 13 sec before opening his parachute. Both skydivers descend at 16 ft/sec with parachutes open. Assume that the skydivers fall freely (no effective air resistance) before their parachutes open.

   (a) At what altitude does Alice's parachute open?

   (b) At what altitude does Bob's parachute open?

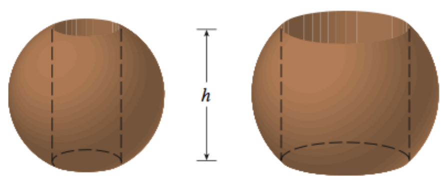
   (c) Which skydiver lands first?

5. **Doing math in wine cellar.** Some of the pioneers of calculus, such as Kepler and Newton, were inspired by the problem of finding the volumes of wine barrels. (In fact Kepler published a book Stereometria doliorum in 1715 devoted to methods for finding the volumes of barrels.) They often approximated the shape of the sides by parabolas.

   (a) A barrel with height h and maximum radius R is constructed by rotating about the x-axis the parabola $y = R - cx^2$, for $-h/2 \leq x \leq h/2$, where c is a positive constant. Find the radius of each end of the barrel.

   (b) Find the volume of the barrel.

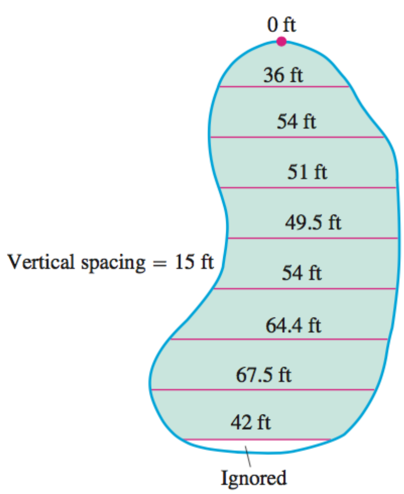
6. **Napkin rings** Suppose you make napkin rings by drilling holes with different diameters through two wooden balls (which also have different diameters). You discover that both napkin rings have the same height h, as shown in the figure.



   (a) Guess which ring has more wood in it.

   (b) Check your guess: Use cylindrical shells to compute the volume of a napkin ring created by drilling a hole with radius r through the center of a sphere of radius R and express the answer in terms of h.

7. **New parking lot.** To meet the demand for parking, your town has allocated the area shown here.



As the town engineer, you have been asked by the town council to find out if the lot can be built for \$10,000. The cost to clear the land will be \$0.10 a square foot, and the lot will cost \$2.00 a square foot to pave. Can the job be done for \$10,000? Use a lower sum estimate to see. (Answers may vary slightly, depending on the estimate used.)

8. **Shoveling dirt.** You sling a shovelful of dirt up from the bottom of a hole with an initial velocity of 32 ft/sec. The dirt must rise 17 ft above the release point to clear the edge of the hole. Is that enough speed to get the dirt out, or had you better duck?