Syllabus for Preliminary Exam on
Theory of Functions of Real Variables

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Prelim will consist of stating definitions listed in Part I, providing formulation and proofs of theorems and statements listed in Part II, and problems testing the knowledge, reasoning, logic, and problem solving skills based on the notions and facts listed below.

Part I - Definitions

You need to be able to state definitions of any of the following notions.

Semester I

Basic knowledge

1. Complement of a set; difference of two sets; symmetric difference; union; intersection of family of sets; disjoint family of sets; power set; Cartesian product of sets.
2. Monotonically increasing/decreasing sequence of sets: $A_i \uparrow$, and $A_i \downarrow$.
3. Relation, equivalence relation, equivalence class.
4. Continuous, uniformly continuous, Lipschitz continuous real-valued function on a real line.

Families of sets

5. Algebra, $\sigma-$algebra.
6. Measurable space, measurable set.
7. $\sigma-$algebra generated by a collection of subsets.
8. Borel $\sigma-$algebra; Borel (measurable) sets.
9. Monotone class

Measures

10. Measure on a set with a $\sigma-$algebra on it; measure space.

11. Finite measure; $\sigma-$finite measure; probability measure; complete measure.

12. Outer measure

13. μ^*-measurable sets.

14. Lebesgue-Stieltjes measure; Lebesgue measure.

15. Premeasure (measure on algebra)

Measurable functions

16. A measurable real- (or complex-) valued function on a measure space.

17. Functions $f = g$ almost everywhere.

Lebesgue integral

18. Characteristic function; simple functions.

19. Lebesgue integral of a simple function, non-negative function, measurable real-valued function, measurable complex-valued function.

20. Integrable function.

Riemann integral

21. Partition of an interval; mesh of a partition

22. Riemann sum for a bounded function on an interval

23. Riemann integrable function; Riemann integral of a function.

Types of convergence of a sequence of functions

24. For a sequence of measurable functions: almost everywhere convergence; convergence in measure; L^p convergence; almost uniform convergence.

Product and signed measures

25. Product $\sigma-$algebra; product measure.

26. Signed measure.

27. Positive set; negative set; null set.

28. Mutually singular measures.

Radon-Nikodym theorem

29. A measure ν is absolutely continuous with respect to a measure μ.

30. Radon-Nikodym derivative of a measure ν with respect to a measure μ.

Differentiation

31. Derivative, anti-derivative of a real-valued function.
32. Bounded variation of a real-valued function on an interval.
33. Absolutely continuous real-valued function on a real line.

Semester II

Sets

1. The closure of a set.
2. The interior of the set.
3. Dense set.
4. Nowhere dense set.

L^p spaces

5. Norm on a vector space. Normed vector space.
6. Metric.
7. Convergent sequence of points in a metric space.
8. Cauchy sequence in a metric space.
9. Complete set of a metric space.
10. Banach space.
11. Separable metric space.
12. Convergent series in a normed vector space.
13. Absolutely convergent series in a normed vector space.
14. L^p-norm on a σ-finite measure space, for $1 \leq p < \infty$.
15. L^∞-norm on a σ-finite measure space.
16. L^p-space, $1 \leq p \leq \infty$.
17. Convolution of two measurable functions.

Fourier transform

18. The Fourier transform of a function $f : \mathbb{R}^n \to \mathbb{C}$

Banach spaces

19. Linear functional on a metric space X.
20. Linear map from a normed linear space X to a normed linear space Y.
21. Positive linear functional on the space of continuous functions $C(X)$.
22. Bounded linear functional on a metric space X.
23. Bounded linear operator.
24. A dual X^* of a metric space X.
25. Fourier transform of a complex-valued function $f \in L^1(\mathbb{R}^n)$.
26. A set of the first category.
27. A set of a second category.
28. An open map between normed vector spaces.
29. Quotient space X/M, for a subspace M of a normed vector space X.
30. A graph of an operator.

Hilbert spaces

31. An inner product.
32. A Hilbert space.
33. Subspace, closed subspace of an inner product space.
34. A convex set of a vector space.
35. Two orthogonal vectors in a Hilbert space.
36. An orthogonal complement of a set in a Hilbert space.
37. Orthogonal projections of a vector.
38. Hilbert-Schmidt norm of a matrix.
39. Sobolev space.
40. (orthogonal) direct sum of two orthogonal closed subspaces of a Hilbert space.
41. An orthogonal/orthonormal subset of a Hilbert space
42. A complete set of orthonormal vectors in a Hilbert space.

43. A basis of a Hilbert space.
44. Fourier series/Fourier coefficients of a function in $L^2[0, 2\pi)$.
45. Trigonometric polynomial.
46. Dirichlet kernel.

Topology

47. Topology/ topological space
48. Open, closed sets in a topological space.
49. Discrite, trivial topologies
50. Interior point; limit point; isolated point.
51. Interior of a set; closure of a set; boundary of a set.
52. Subspace of a topological space; relatively open sets; relative topology.
53. Weaker and stronger topologies between two topologies
54. Open base; subbase; open base at a point
55. Product topology
56. Dense set; nowhere dense set; separable set.
57. Second countable topological space.
58. First countable topological space.
59. Convergence of points in a topological space; subsequential or cluster point of a sequence.
60. Directed set
61. Net; convergence of a net.
62. Continuous function; open function; homeomorphism between topological spaces.
63. Over cover for a set in a topological space; compact set.
64. Finite intersection property
65. Precompact set; $\sigma-$ compact; countable compact; sequentially compact sets
66. Basic and subbasic open cover for a set
67. Bounded set of a metric space
68. ϵ-net; totally bounded set

69. A uniformly continuous function between metric spaces
70. Isometry between metric spaces
71. A completion of a metric space
72. T_1 space; Hausdorff space; completely regular space; normal space

Spectral Theory

73. A spectrum of a bounded linear operator on a complex Hilbert space
74. The resolvent set
75. Spectral radius of an operator
76. Eigenvector; eigenvalue
77. An adjoint of a bounded linear operator
78. Symmetric operator (or Hermitian, or self-adjoint)
79. Compact operator on a separable Hilbert space

Part II - Theorems with proofs

You need to know the formulation and the proof of each of the theorem/statement posted below.

Semester I

Families of sets

1. The intersection of any number of σ-algebras is a σ-algebra.

2. Let $X = \mathbb{R}$, then the Borel σ-algebra is generated by each of the following collection of sets: 1) finite open intervals; 2) finite closed intervals; 3) finite half open half closed intervals; 4) open rays $(a, \infty), a \in \mathbb{R}$.

3. The monotone class theorem.

Measures

4. For a measure space $(X, \mathcal{A}, \mu)$, if $A_i \in \mathcal{A}$ and $A_i \uparrow A$, then $\mu(A) = \lim_{n\to\infty} \mu(A_n)$.

5. For a measure space $(X, \mathcal{A}, \mu)$, if $A_i \in \mathcal{A}$, $A_i \downarrow A$, and $\mu(A_1) < \infty$, then $\mu(A) = \lim_{n\to\infty} \mu(A_n)$.

6. Caratheodory's theorem

7. Every Borel set on $\mathbb{R}$ is Lebesgue-Stieltjes measurable.

8. Suppose $A \subset [0, 1]$ is a Lebesgue set with m being a Lebesgue measure. Prove that given $\epsilon > 0$, there exists an open set G containing A, such that $m(G - A) < \epsilon$.

9. Suppose $A \subset [0, 1]$ is a Lebesgue set with m being a Lebesgue measure. Prove that there exists a set H containing A, which is the countable intersection of a decreasing sequence of open sets, such that $m(H - A) = 0$.

10. Caratheodory's Extension Theorem

Measurable functions

11. If f and g are measurable real-valued functions, then so are $f + g$, $f - g$, cf, fg, $\max(f, g)$, $\min(f, g)$. If f_n are measurable, then so are $\sup f_n$, $\inf f_n$, $\liminf f_n$, $\limsup f_n$.

12. If $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is Borel measurable.

13. Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to \mathbb{R}$ be an $\mathcal{A}$ measurable function. If C is in the Borel σ-algebra on $\mathbb{R}$, then $f^{-1}(C) \in \mathcal{A}$.

14. Suppose f is a non-negative and measurable function on X. Then there exists a sequence of non-negative measurable simple functions s_n such that

$$0 \leq s_1 \leq s_2 \leq \cdots \leq f \ ,$$

and $s_n \to f$ pointwise on X, and $s_n \to f$ uniformly on any set where f is bounded.

15. Lusin's Theorem

The Lebesgue integral

16. For two non-negative simple functions ϕ and ψ, $\int(\phi+\psi)d\mu = \int \phi d\mu + \int \psi d\mu$.

17. Monotone Convergence Theorem

18. If f is non-negative measurable function, then $\int f d\mu = 0$ iff $f = 0$ a.e. .

19. Linearity of the Lebesgue integral

20. Suppose $f_n \geq 0$ are measurable functions. Then $\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n$.

21. If f is integrable, then $\left|\int f\right| \leq \int |f|$.

Limit theorems

22. Fatou's lemma

23. Dominated Convergence Theorem

24. Suppose f_n is a sequence of integrable functions such that $\sum_n \int |f_n| < \infty$. Then the series $f(x) = \sum_n f_n(x)$ converges a.e., f is integrable and $\int f = \sum_n \int f_n$.

25. Borel-Cantelli lemma

Properties of Lebesgue integrals

26. Suppose f is real-valued and integrable, and for every measurable set A we have $\int_A f d\mu = 0$. Then $f = 0$ a.e.

27. For a Lebesgue measure m and any $a \in \mathbb{R}$, suppose that $f : \mathbb{R} \to \mathbb{R}$ is integrable and $\int_a^x f(y)dy = 0$ for any x. Then $f = 0$ a.e.

28. Suppose f is Lebesgue measurable real-valued integrable function on $\mathbb{R}$. Let $\epsilon > 0$. Then there exists a continuous function g with compact support s.t. $\int |f - g| < \epsilon$.

Riemann integral

29. Every monotone function on $[a, b]$ is Riemann integrable.

30. Every continuous function on $[a, b]$ is Riemann integrable.

31. If a bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then f is continuous a.e. with respect to a Lebesgue measure. In this case, f is Lebesgue integrable and $R(f) = \int f$.

Types of convergence

32. Suppose μ is a finite measure. Then if $f_n \to f$ a.e., then $f_n \to f$ in measure.

33. If $f_n \to f$ in measure, then there exists a subsequence n_j such that $f_{n_j} \to f$ a.e.

34. If $f_n \to f$ in L^p, then $f_n \to f$ in measure.

35. Chebyshev's inequality.

36. Egorov's theorem.

Product measures

37. Fubini theorem: every part, including full proofs for characteristic function.

38. Prove that $\lim_{b\to\infty} \int_0^b \frac{\sin x}{x} dx = \frac{\pi}{2}$.

Signed measures

39. Let μ be a signed measure. Let E be measurable with $\mu(E) < 0$. Then there exists a measurable subset F of E that is a negative set with $\mu(F) < 0$.

40. Hahn decomposition theorem.

41. Jordan decomposition theorem.

Radon-Nikodym theorem

42. Let ν be a finite measure. Then $\nu << \mu$ if and only if for all $\epsilon > 0$ there exists $\delta > 0$ such that if $\mu(A) < \delta$ then $\nu(A) < \epsilon$.

43. Radon-Nikodym theorem (you may use supporting lemma without proof after stating it in full).

44. Lebesgue decomposition theorem.

Differentiation

45. Hardy-Littlewood inequality (you may use covering lemma without proof after stating it in full).

46. If f is locally integrable, then

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} f(t)dt \to f(x) \quad \text{a.e.}$$

as $r \to 0$.

47. For a.e. x

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(t) - f(x)|dt \to 0$$

as $r \to 0$.

48. For an integrable function $f : \mathbb{R} \to \mathbb{R}$, the antiderivative F is differentiable a.e., and $F'(x) = f(x)$ a.e.

49. Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and right continuous. Then F is differentiable a.e., it is locally integrable, and for $a < b$, $\int_a^b F'(x)dx \le F(b) - F(a)$.

50. Let $F : \mathbb{R} \to \mathbb{R}$ be increasing. Then F is differentiable a.e., and for $a < b$, $\int_a^b F'(x)dx \le F(b) - F(a)$.

51. If f is of bounded variation on $[a, b]$, then $f = v_f^+ - v_f^-$, where $v_f^\pm$ are increasing functions on $[a, b]$.

52. If f is absolutely continuous, it is of bounded variation.

53. If f is absolutely continuous, and $f = v_f^+ - v_f^-$, then $v_f^\pm$ are absolutely continuous.

54. If F is absolutely continuous, then F is differentiable a.e., it is integrable, and for $a < b$, $\int_a^b F'(x)dx = F(b) - F(a)$.

Semester II

If not restricted further, assume that any L^p space is defined on a σ-finite measure space.

Fourier transform is defined for $f : \mathbb{R}^n \to \mathbb{C}$, $f \in L^1$ as

$$\hat{f}(u) = \int e^{iu\cdot x} f(x)dx ,$$

where $u \cdot x$ is the inner product.

L^p spaces

1. A normed vector space X is complete if and only if every absolutely convergent series in X converges to an element in X.
2. Hölder's inequality
3. Minkowski's inequality
4. L^p is a Banach space for $1 \le p \le \infty$.
5. The space of continuous functions with compact support on $\mathbb{R}$, C_c, is dense in $L^p(\mathbb{R})$, for $1 \le p < \infty$.
6. In a finite measure space X, $L^r \subset L^s$ for $1 \le s < r \le \infty$.
7. In a probability measure space, assume that for some $1 \le r < \infty$, $\|f\|_r < \infty$. Then $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.
8. If $f, g \in L^1(\mathbb{R})$, then $fg \in L^1(\mathbb{R})$, and $\|f * g\|_1 \le \|f\|_1\|g\|_1$.
9. If $1 < p \le \infty$, $f \in L^1$, $g \in L^p$, then $fg \in L^p$, and $\|f * g\|_p \le \|f\|_1\|g\|_p$.
10. Suppose $A, B \subset X$ are such that $0 < m(A) < \infty$ and $0 < m(B) < \infty$. Then the convolution of the characteristic functions $\chi_A * \chi_B$ is continuous.
11. Suppose $A \subset \mathbb{R}$, and $0 < m(A) < \infty$. Then $A - A = \{x - y : x, y \in A\}$ contains an interval containing the origin.

12. $L^{\infty}(\mathbb{R}^n)$ is not separable. (proof for $n = 1$)

13. $L^p(\mathbb{R}^n)$ is separable, $1 \leq p < \infty$. (proof for $n = 1$)

14. If $1 < p < \infty$, $p^{-1} + q^{-1} = 1$, and $f \in L^p$, then $\|f\|_p = \sup\{\int fg \, \mathrm{d}\mu : \|g\|_q \leq 1\}$.

15. Suppose $1 < p < \infty$, $p^{-1} + q^{-1} = 1$, and $g \in L^q$. Define $H_g(f) = \int fg$ for any $f \in L^p$. Then H_g is a bounded linera functional on L^p and $\|H_g\| = \|g\|_q$.

16. Dual of L^p can be identified with L^q. Suppose $1 < p < \infty$, $p^{-1} + q^{-1} = 1$, and H be a real-valued bounded linear functional on L^p. Then there exists $g \in L^q$, s.t. $H = H_g$.

Fourier transform

17. Suppose $f \in L^1(\mathbb{R}^n)$ and $x_j f(x) \in L^1$. Then $\frac{\partial \hat{f}}{\partial u_j}(u) = i \int e^{iu\cdot x} x_j f(x) \mathrm{d}x$.

18. If $f : \mathbb{R} \to \mathbb{R}$ is integrable, absolutely continuous, then the Fourier transform of the derivative, $\widehat{f'}(u) = -iu\hat{f}(u)$.

19. If $f, g \in L^1$, then $\widehat{f * g} = \hat{f}\hat{g}$.

20. Suppose $f_n : \mathbb{R}^n \to \mathbb{R}$ is given by $f_n(x) = (2\pi)^{-n/2} e^{-|x|^2/2}$, then $\widehat{f_n}(u) = e^{-|u|^2/2}$.

21. Suppose $\phi \in L^1$ and $\int \phi dx = 1$. Let $\phi_\delta(x) = \delta^{-n}\phi(x/\delta)$.
    If g is continuous with compact support, then

    (a) $g * \phi_n \to g$ pointwise as $\delta \to 0$;

    (b) $g * \phi_n \to g$ in L^1 as $\delta \to 0$.

    If $f \in L^1$, then $\|f * \phi_n - f\|_1 \to 0$ as $\delta \to 0$.

22. If $f, \hat{f} \in L^1$, then $f(y) = (2\pi)^{-n} \int e^{-iu\cdot y} \hat{f}(u) du$.

23. Plancherel theorem

Riesz representation theorem

24. Partition of unity for compact metric spaces.

25. Riesz representation theorem for positive linear functionals.

26. Suppose I is a bounded linear functional on $C(X)$. Then there exist positive bounded linear functionals, J and K, s.t. $I = J - K$.

27. Riesz representation theorem for bounded linear functionals.

Banach spaces

28. For a linear operator $T : X \to Y$, the following are equivalent:

    (a) T is bounded

(b) T is continuous

(c) T is continuous at one point in X.

29. If Y is a Banach space, then the space of all bounded linear operators, $B(X, Y)$ is a Banach space.

30. Hahn-Banach theorem

31. Baire's category theorem

32. Banach-Steinhaus theorem (uniform boundedness theorem)

33. Open mapping theorem

34. Closed graph theorem

35. For a Banach space X, if its dual, X^*, is separable, then X is separable.

Hilbert space

36. Cauchy-Schwarz inequality in Hilbert space.

37. Triangle inequality for a norm induced by an inner product in a Hilbert space.

38. A norm induced by an inner product in a Hilbert space is continuous.

39. In a Hilbert space H, for a fixed $y \in H$, the function $x \mapsto \langle x, y \rangle$ is continuous.

40. Parallelogram Law.

41. Each non-empty closed convex subset of a Hilbert space has a unique element of smallest norm.

42. Let M be a closed subspace of a Hilbert space H, such that $M \neq H$. Then the orthogonal complement, $M^\perp$ contains a non-zero element.

43. If a norm in a normed vector space satisfies the polarization formula, then this normed vector space is an inner product space with the norm induced by this inner product.

44. Riesz representation theorem for a bounded linear function on a Hilbert space.

45. Gram-Schmidt procedure

46. Bessel's inequality

47. Suppose $U = \{u_\alpha\}$ are orthonormal in a Hilbert space H. Then the following are equivalent:

    (a) $U^\perp = \{0\}$

    (b) Parseval's identity: $\forall x \in H: \ \|x\|^2 = \sum_\alpha |\langle x, u_\alpha \rangle|$

    (c) $\forall x \in H: \ x = \sum_\alpha \langle x, u_\alpha \rangle u_\alpha$.

48. Every Hilbert space has an orthonormal basis (use may use Zorn's Lemma without proof).

49. Any orthogonal set in L^2 is countable.

50. A Hilbert space is separable if and only if it has a countable orthonormal basis.

51. Suppose $f \in C[0, 2\pi]$ and $f(0) = f(2\pi)$. Let $\epsilon > 0$. Then there exists a trigonometric polynomial g s.t. $\sup\{|f(x) - g(x)| < \epsilon \ : \ 0 \leq x \leq 2\pi\}$.

52. The set of trigonometric polynomials is dense in $L^2[0, 2\pi)$.

53. Riemann-Lebesgue theorem

54. Suppose f is bounded and measurable on $[0, 2\pi)$. Extend f periodically to $\mathbb{R}$. Let $x \in [0, 2\pi)$. Suppose there exists $K > 0$ s.t. for all h: $|f(x+h) - f(x)| <\leq K|h|$. Then $\lim_{N\to\infty} S_N f(x) = f(x)$, where $S_N f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^{N} c_n e^{inx}$, where $c_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(y) e^{-iny} dx$ is a Fourier coefficient of f.

55. Let $f \in L^2[0, 2\pi)$. Then of all linear combinations $\sum_{n=-N}^{N} a_n u_n$, $a_n \in \mathbb{C}$ with fixed N, the optimal L^2-approximation is given by $S_N f = \sum_{n=-N}^{N} c_n u_n$. Here $u_n = \frac{1}{\sqrt{2\pi}} e^{inx}$.

56. Riesz-Fischer theorem

Topology

57. For a topological space $(X, \mathcal{T})$, if $A \subset X$, then $\bar{A} = \cap\{F \ : \ F \text{ is closed, } A \subset F\}$.

58. Let X be a metric space. Then X is second countable if and only if it is separable.

59. Let E be a subset of a topological space. If there is a net consisting of infinitely many points in E that converge to y, then y is a limit point of E.

60. Let E be a subset of a topological space. If y is a limit point of E, then there is a net in E that converge to y.

61. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, and A be a subbase for Y. Then for a function $f : X \to Y$, if for any set $G \in S$ we have $f^{-1}(G) \in \mathcal{T}$, then f is continuous.

62. A closed subset of a compact set in a topological space is compact.

63. Continuous function between topological spaces maps compact sets into compact sets.

64. A topological space is compact if and only if any collection of closed sets with the finite intersection property has non-empty intersection.

65. If every basic open cover for a set A in a topological space has a finite subcover, then A is compact.

66. If a subset of a metric space is compact, then it is closed and bounded.

67. If a subset of a metric space is sequentially compact, then it is compact.

68. A subset of a metric space is compact if and only if it is totally bounded and complete.

69. If a functions between a compact metric space and a metric space is continuous, then it is uniformly continuous.

70. Completely regular spaces are Hausdorff spaces.

71. Let X be a Hausdorff space, $F \subset X$ compact, and $x \notin F$. Then there exist disjoint open sets G and H, s.t. $x \in G$ and $F \subset H$.

72. Compact subsets of Hausdorff spaces are closed.

73. A compact Hausdorff space is normal.

74. Urysohn's lemma.

Spectral Theory

75. If B is a bounded linear operator on a Hilbert space H with $\|B\| < 1$, then $I - B$ is invertible and

$$(I - B)^{-1} = \sum_{i=0}^{\infty} B^i \ .$$

76. If A is an invertible bounded linear operator on a Hilbert space H, and B is a bounded linear operator on H with $\|B\| < 1/\|A^{-1}\|$, then $A - B$ is invertible.

77. If A is a bounded linear operator on a Hilbert space H, then the spectrum of A, $\sigma(A)$, is a closed and bounded subset of $\mathbb{C}$, and the spectral radius $f(A) \leq \|A\|$.

78. The adjoint operator always exists (for a bounded operator on a Hilbert space).

79. For a bounded Hermitian operator,
1)$\langle Ax, x\rangle$ is real for all $x \in H$;
2) the function $x \mapsto \langle Ax, x\rangle$ is not identically 0 unless $A = 0$;
3) $\|A\| = \sum_{\|x\|=1} |\langle Ax, x\rangle|$.

80. If K and L are compact operators and $c \in \mathbb{C}$, then $cK + L$ is a compact operator.

81. If L is a bounded linear operator on a Hilbert space H, and K is a compact operator, then KL and LK are compact operators.

82. If K_n are compact operators and $\lim_{n\to\infty} \|K_n - K\| = 0$, then K is a compact operator.

83. Let S_1 and S_2 be two subsets of a Hilbert space whose closure is compact. Then the closure of the sum $S_1 + S_2 = \{a + b \ : \ a \in S_1, b \in S_2\}$ is compact.

84. Let K be a compact Hermitian linear operator on a separable Hilbert space H. There exist a sequence $\{z_n\} \subset H$ and $\{\lambda_n\} \subset \mathbb{R}$ such that

    (a) $\{z_n\}$ is an orthonormal basis for H;

    (b) each z_n is an eigenvector with eigenvalue λ_n;

    (c) for each $\lambda_n \neq 0$, the dimension of the linear space $\{x \in H \ : \ Kx = \lambda_n x\}$ is finite;

    (d) the only limit point, if any, of $\{\lambda_n\}$ is 0; if there are infinitely many distinct eigenvalues, then 0 is a limit point of $\{\lambda_n\}$.