Preliminary Exam on the Theory of Functions of Real Variables

May 2023

Full Name (First and Last) ____________________________________________________________

Student ID______________________________________________________________

You are required to adhere to the UH Academic Honesty policy. No notes, books, phones, tables, calculators, computers, or any other help is allowed on the exam.

You are required to fill in, complete and proof all prompts in the text.

The exam is 3 hours long.

GOOD LUCK!
Problem 1
Suppose $\mu^*$ is an outer measure on $X$, $A$ a subset of $X$, and for each $\epsilon > 0$ there exists $B$ such that $A \subset B$, $B$ is $\mu^*$-measurable, and $\mu^*(B - A) < \epsilon$. Prove that $A$ is $\mu^*$-measurable.

Definitions
State the definition: $\mu^*$ is an **outer measure** if and only if

State the definition: A set $A$ is $\mu^*$-**measurable** if and only if

Proof
Problem 2

Suppose $f_n, g_n, f, g$ are integrable functions, such that $g_n \geq 0$ and

1. $f_n \to f$ a.e. and $g_n \to g$ a.e.,
2. $|f_n| \leq g_n$ for each $n$,
3. $\int g_n \to \int g$.

Show that $\int f_n \to \int f$.

Definitions

State the definition: $f_n \to f$ a.e. if and only if

State Fatou’s Lemma:

Proof of Problem 2
Problem 3
Suppose $\mu$ and $\nu$ are two finite positive measures such that $\nu$ is absolutely continuous with respect to $\mu$. Let $\rho = \mu + \nu$.
Prove that if $f = \frac{d\mu}{d\rho}$ and $g = \frac{d\nu}{d\rho}$, then $f$ is strictly positive for almost every $x$ with respect to $\mu$, $f + g = 1$ for almost every $x$ with respect to $\rho$, and $d\nu = (g/f)d\mu$.

Definitions
State the definition: A measure $\nu$ is absolutely continuous with respect to a measure $\mu$, $\nu \ll \mu$, if and only if

State the Radon-Nikodym theorem:

What is the Radon-Nikodym derivative of $\nu$ with respect to $\mu$, denoted as $\frac{d\nu}{d\mu}$?

In the above problem 3, why Radon-Nikodym derivatives of $\mu$ and $\nu$ with respect to $\rho$ exist?

Proof
First, show that $f$ is strictly positive a.e. with respect to $\mu$. 
Proof of Problem 3

Show that \( f(x) + g(x) = 1 \) for a.e. \( x \) with respect to \( \rho \).

Show that the Radon-Nikodym derivative of \( \nu \) with respect to \( \mu \) is \( \frac{d\nu}{d\mu} = gf^{-1} \).
Prove that $f$ is Lipschitz continuous with constant $M$ if and only if $f$ is absolutely continuous and $|f'| \leq M \text{ a.e.}$.

**Definitions**

State the definition: $f : \mathbb{R} \to \mathbb{R}$ is **Lipschitz continuous** with constant $M$ if and only if

State the definition: $f : \mathbb{R} \to \mathbb{R}$ is **absolutely continuous** if and only if

**Proof**

$(\Rightarrow)$: Suppose that $f$ is Lipschitz continuous.
Proof of Problem 4 cont.

(⇐): Suppose that \( f \) is absolutely continuous and \( |f'| \leq M \text{ a.e.} \). Show that \( f \) is Lipschitz continuous.
Problem 5
Suppose $1 \leq r < p < s < \infty$ and $(X, \mathcal{A}, \mu)$ is a $\sigma$-finite measure space. Prove that $L^r(X) \cap L^s(X) \subset L^p(X)$.

Definition

Complete the definition: A function $f \in L^p(X)$ if and only if

Proof of Problem 5
Problem 6

Suppose $X$ and $Y$ are Banach spaces, and $T : X \to Y$ is a bijective bounded linear operator. Prove that $T^{-1}$ is a bounded linear operator.

Open Mapping Theorem

State the definition: $(X, \| \cdot \|)$ is a complete normed vector space (i.e. Banach) if and only if...

State the definition: $L : X \to Y$ is bounded if and only if...

State the Open Mapping Theorem

State the definition: $L : X \to Y$ is open if and only if...

State the definition: In a normed vector space $(X, \| \cdot \|)$ a set $U$ is open if and only if...
Proof of Problem 6
Problem 7

Prove Heisenberg’s inequality: for \( f \in L^2(\mathbb{R}) \), and any \( a, b \in \mathbb{R} \),

\[
\left( \int (x-a)^2 f(x)^2 \, dx \right) \left( \int (u-b)^2 \hat{f}(u)^2 \, du \right) \geq \frac{\pi}{2} \left( \int f(x)^2 \, dx \right)^2.
\]

Definitions

For a complex-valued function \( f \), define its \textbf{Fourier transform} \( \hat{f} : \mathbb{R} \rightarrow \mathbb{C} \), by

\[
\hat{f}(u) = 
\]

The Fourier transform of a derivative is:

\[
\hat{f}'(u) = 
\]

State the Plancherel’s formula:

Let \( f_a(x) = f(x + a) \), then its Fourier transform is,

\[
\hat{f}_a(u) = 
\]

You may use the following formula without proof:

\[
\left( \int |f(x)|^2 \, dx \right)^2 \leq 4 \left( \int |xf(x)|^2 \, dx \right) \left( \int |f'(x)|^2 \, dx \right).
\]

Proof

First, prove the Heisenberg’s inequality for \( a = b = 0 \), then generalize to any constants.
Problem 8

Let $f$ map a topological space $(X, T)$ into a topological space $(Y, U)$. Prove that $f$ is continuous if and only if whenever $x \in X$ and $G$ is an open set in $Y$ containing $f(x)$, there exists an open set $H$ in $X$ such that $f(y) \in G$ whenever $y \in H$.

Definitions

State the definition: A map $f$ between topological spaces $(X, T)$ and $(Y, U)$ is continuous if and only if

Proof

$(\Rightarrow)$: Suppose that $f$ is continuous, then

$(\Leftarrow)$: Suppose that $f$ satisfies the condition in the statement, show that $f$ is continuous:
Problem 9
Give an example of a bounded linear operator $A$ on $l^2(\mathbb{R})$ which has 0 in its spectrum, but not as an eigenvalue.

Definitions

State the definition: Let $A$ be a bounded linear operator on a Hilbert space $H$ over complex numbers. A **spectrum**, $\sigma(A)$, of $A$ is

State the definition: A vector $x \in H$ is an **eigenvector** of an operator $A$ corresponding to an **eigenvalue** $\lambda$ if and only if

Why your example below does not contradict Problem 6?

Proof

Define operator $A$. Then show that 1) it is bounded; 2) $0 \in \sigma(A)$; 3) $0$ is not an eigenvalue.
State which problem you are working on here. If you would like this page (or a part of it) to be graded, write "See scratch paper" on the previous page for the corresponding problem.
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