# Preliminary Exam on the Theory of Functions of Real Variables

May 2023

**Full Name (First and Last)** ______________________________________________________

**Student ID**______________________________________________________

You are required to adhere to the UH Academic Honesty policy. No notes, books, phones, tables, calculators, computers, or any other help is allowed on the exam.

You are **required** to fill in, complete and proof all prompts in the text.

The exam is 3 hours long.

GOOD LUCK!

## Problem 1

Suppose $\mu^*$ is an outer measure on $X$, $A$ a subset of $X$, and for each $\epsilon > 0$ there exists $B$ such that $A \subset B$, $B$ is $\mu^*$-measurable, and $\mu^*(B - A) < \epsilon$. Prove that $A$ is $\mu^*$-measurable.

## Definitions

State the definition: $\mu^*$ is an **outer measure** if and only if

State the definition: A set $A$ is $\mu^*$**-measurable** if and only if

## Proof

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## Problem 2

Suppose $f_n, g_n, f, g$ are integrable functions, such that $g_n \geq 0$ and

1. $f_n \to f$ a.e. and $g_n \to g$ a.e.,
2. $|f_n| \leq g_n$ for each $n$,
3. $\int g_n \to \int g$.

Show that $\int f_n \to \int f$.

## Definitions

State the definition: $f_n \to f$ **a.e.** if and only if

State Fatou's Lemma:

## Proof of Problem 2

## Proof of Problem 2

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**Problem 3**

Suppose $\mu$ and $\nu$ are two finite positive measures such that $\nu$ is absolutely continuous with respect to $\mu$. Let $\rho = \mu + \nu$. Prove that if $f = \frac{d\mu}{d\rho}$ and $g = \frac{d\nu}{d\rho}$, then $f$ is strictly positive for almost every $x$ with respect to $\mu$, $f + g = 1$ for almost every $x$ with respect to $\rho$, and $d\nu = (g/f)d\mu$.

**Definitions**

State the definition: A measure $\nu$ is **absolutely continuous** with respect to a measure $\mu$, $\nu \ll \mu$, if and only if

State the Radon-Nikodym theorem:

What is the Radon-Nikodym derivative of $\nu$ with respect to $\mu$, denoted as $\frac{d\nu}{d\mu}$?

In the above problem 3, why Radon-Nikodym derivatives of $\mu$ and $\nu$ with respect to $\rho$ exist?

**Proof**

First, show that $f$ is strictly positive a.e. with respect to $\mu$.

**Proof of Problem 3**

Show that $f(x) + g(x) = 1$ for a.e. $x$ with respect to $\rho$.

Show that that the Radon-Nikodym derivative of $\nu$ with respect to $\mu$ is $\frac{d\nu}{d\mu} = gf^{-1}$.

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## Problem 4

Prove that $f$ is Lipschitz continuous with constant $M$ if and only if $f$ is absolutely continuous and $|f'| \leq M$ a.e..

## Definitions

State the definition: $f : \mathbb{R} \to \mathbb{R}$ is **Lipschitz continuous** with constant $M$ if and only if

State the definition: $f : \mathbb{R} \to \mathbb{R}$ is **absolutely continuous** if and only if

## Proof

$(\Rightarrow)$: Suppose that $f$ is Lipschitz continuous.

**Proof of Problem 4 cont.**

($\Leftarrow$): Suppose that $f$ is absolutely continuous and $|f'| \leq M$ a.e.. Show that $f$ is Lipschitz continuous.

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**Problem 5**

Suppose $1 \leq r < p < s < \infty$ and $(X, \mathcal{A}, \mu)$ is a $\sigma$-finite measure space. Prove that $L^r(X) \cap L^s(X) \subset L^p(X)$.

**Definition**

Complete the definition: A function $f \in L^p(X)$ if and only if

**Proof of Problem 5**

**Problem 6**

Suppose $X$ and $Y$ are Banach spaces, and $T : X \to Y$ is a bijective bounded linear operator. Prove that $T^{-1}$ is a bounded linear operator.

**Open Mapping Theorem**

State the definition: $(X, \|\cdot\|)$ is a **complete** normed vector space (i.e. Banach) if and only if

State the definition: $L : X \to Y$ is **bounded** if and only if

State the Open Mapping Theorem

State the definition: $L : X \to Y$ is **open** if and only if

State the definition: In a normed vector space $(X, \|\cdot\|)$ a set $U$ is **open** if and only if

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## Proof of Problem 6

**Problem 7**

Prove Heisenberg's inequality: for $f \in L^2(\mathbb{R})$, and any $a, b \in \mathbb{R}$,

$$\left(\int (x-a)^2 f(x)^2 dx\right)\left(\int (u-b)^2 \widehat{f}(u)^2 du\right) \geq \frac{\pi}{2}\left(\int f(x)^2 dx\right)^2 .$$

**Definitions**

For a complex-valued function $f$, define its **Fourier transform** $\widehat{f} : \mathbb{R} \to \mathbb{C}$, by

$\widehat{f}(u) =$

The Fourier transform of a derivative is:

$\widehat{f'}(u) =$

State the Plancherel's formula:

Let $f_a(x) = f(x+a)$, then its Fourier transform is,

$\widehat{f_a}(u) =$

You may use the following formula without proof:

$$\left(\int |f(x)|^2 dx\right)^2 \leq 4\left(\int |xf(x)|^2 dx\right)\left(\int |f'(x)|^2 dx\right) .$$

**Proof**

First, prove the Heisenberg's inequality for $a = b = 0$, then generalize to any constants.

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**Proof of Problem 7 cont.**

**Problem 8**

Let $f$ map a topological space $(X, \mathcal{T})$ into a topological space $(Y, \mathcal{U})$. Prove that $f$ is continuous if and only if whenever $x \in X$ and $G$ is an open set in $Y$ containing $f(x)$, there exists an open set $H$ in $X$ such that $f(y) \in G$ whenever $y \in H$.

**Definitions**

State the definition: A map $f$ between topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ is **continuous** if and only if

**Proof**

($\Rightarrow$): Suppose that $f$ is continuous, then

($\Leftarrow$): Suppose that $f$ satisfies the condition in the statement, show that $f$ is continuous:

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## Proof of Problem 8 cont.

## Problem 9

Give an example of a bounded linear operator $A$ on $l^2(\mathbb{R})$ which has 0 in its spectrum, but not as an eigenvalue.

## Definitions

State the definition: Let $A$ be a bounded linear operator on a Hilbert space $H$ over complex numbers. A **spectrum**, $\sigma(A)$, of $A$ is

State the definition: A vector $x \in H$ is an **eigenvector** of an operator $A$ corresponding to an **eigenvalue** $\lambda$ if and only if

Why your example below does not contradict Problem 6?

## Proof

Define operator $A$. Then show that 1) it is bounded; 2) $0 \in \sigma(A)$; 3) 0 is not an eigenvalue.

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**Proof of Problem 9 cont.**

**Scratch paper**

State which problem you are working on here. If you would like this page (or a part of it) to be graded, write "See scratch paper" on the previous page for the corresponding problem.

**Name**____________________________________________________

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