RESEARCH STATEMENT
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My research interests are in the fields of Number Theory, Dynamical Systems and Ergodic Theory. In what follows, I summarize my current research work in these areas, the motivation towards undertaking the projects I have been involved in so far and also some of the interesting problems that I look forward to working on next.

Introduction

I have been part of the Dynamical Systems research group during my tenure as a PhD student at the University of Houston.

The overall theme of my research till now has been to study problems related to Diophantine approximation in p-adic solenoids and their interactions with dynamical systems. In order to understand problems in dynamics that are sensitive to arithmetic properties of return times to regions, it is desirable to generalize classical results about rotations on $\mathbb{R}/\mathbb{Z}$ to the setting of rotations on adelic tori. One of the limitations in working with the s-torus, $\mathbb{T} = (\mathbb{R}/\mathbb{Z})^s$, $s \geq 2$ is that we only have countably many isomorphism classes of compact groups. This limitation no longer exists when we work with adeles.

For the first of my two completed projects, we worked on giving an explicit construction of **Bounded Remainder Sets** of all possible volumes, for rotations on higher dimensional adelic tori. The second project pertains to proving a natural generalization of the **Three Gap Theorem** for rotations on adelic tori.

Before we start discussing the projects, I give here a brief introduction to what led to my interest in these areas. I attribute my introduction to this field of study to a course on Topological Dynamics that I audited during the last year of my Masters degree program. From the first semester of my PhD program, I started reading multiple research papers under the guidance of my thesis advisor, Alan Haynes. These papers exposed me to even more ideas from Dynamical Systems and Ergodic Theory.

One of the first papers that I read is authored by Haynes in collaboration with Henna Koivusalo and James Walton (see [11]). Take a linear $\mathbb{R}^d$-action on a higher dimensional torus and consider the return times of this action to some particular region. The collection of points in $\mathbb{R}^d$ that arise from this dynamical construction give us point sets, which we call **cut and project sets**. These sets are widely accepted as mathematical models for physical materials known as quasicrystals. The goal of their paper is to explore the possible existence of cut and project sets that are better than "perfectly ordered". The results of the paper also lead to an understanding of how the study of the existence of these sets is closely related to the **Littlewood Conjecture**, a famous open problem in Diophantine Approximation. This paper introduced me to a technique in tiling theory known as the "cut and project set method". It led me to appreciate the connections that exist between Number Theory, Dynamical Systems and Ergodic Theory.

Here I give an introduction to p-adic analysis and notation. For more detailed background information, I refer the reader to [18, Chapter 1] and [23, Chapter 3]. Additionally, for detailed definitions and basic properties of the adeles or adelic tori, please refer to [4, Section 2] and [3, Section 2].

If $a \neq 0 \in \mathbb{Q}$, then the p-adic absolute value is defined as $|a|_p = p^{-m}$, where $a = p^m \left(\frac{b}{c} \right)$, $b, c \in \mathbb{Z}, p \nmid b, c$, and by convention, $|0|_p = 0$. For any prime number p, we write $\mathbb{Q}_p$ for the field of p-adic numbers, which is the completion of $\mathbb{Q}$ with respect to $| \cdot |_p$. Every element $x \in \mathbb{Q}_p$ can be expressed as a sum of the form

$$x = \sum_{i=N}^{\infty} x_i p^i,$$

where $x_i \in \{0, 1, \ldots, p-1\}$ for all integers $i \geq N$. The ring of p-adic integers $\mathbb{Z}_p$ is the set of all $x \in \mathbb{Q}_p$ with $|x|_p \leq 1$. We use $| \cdot |_\infty$ to denote the usual Archimedean absolute value on $\mathbb{R}$. Let $\mathbb{A}$ denote the topological ring of rational adeles over $\mathbb{Q}$ consisting of all points of the form

$$\boldsymbol{\alpha} = (\alpha_\infty, \alpha_{p_1}, \alpha_{p_2}, \ldots) \in \mathbb{R} \times \prod_p \mathbb{Q}_p,$$

satisfying the condition that $\alpha_p \in \mathbb{Z}_p$ for all but finitely many primes p (the product above is over all prime numbers). Addition and multiplication of elements are defined pointwise and the topology on $\mathbb{A}$ is the restricted product topology with respect to the sets $\mathbb{Z}_p \subseteq \mathbb{Q}_p$. Let $\mathcal{P} = \{p_1, p_2, \ldots\}$ be a non-empty subset of prime numbers. Then, $\mathbb{A}_\mathcal{P}$ denotes the topological ring obtained as a result of the projection of $\mathbb{A}$ onto the places indexed by $\{\infty\} \cup \mathcal{P}$. It is provided with the final topology with respect to this projection. The additive group $\Gamma_\mathcal{P} = \mathbb{Z}[1/p_1, 1/p_2, \ldots]$ can be diagonally embedded into $\mathbb{A}_\mathcal{P}$ by the injective homomorphism $\gamma \mapsto \boldsymbol{\gamma} = (\gamma, \gamma, \gamma, \ldots)$, and we identify $\Gamma_\mathcal{P}$ with its image under this map. $\Gamma_\mathcal{P}$ is a discrete subgroup of $\mathbb{A}_\mathcal{P}$ and we subsequently define the quotient group $X_\mathcal{P} = \mathbb{A}_\mathcal{P}/\Gamma_\mathcal{P}$. This is a connected, compact, metrizable, abelian group. From a topological visualization point of view, the 2-adic solenoid (simplest case of an adelic torus where $\mathcal{P} = \{2\}$), $X_2 = (\mathbb{R} \times \mathbb{Q}_2)/\mathbb{Z}[1/2]$, is homeomorphic to the Smale-Williams solenoid.

Bounded Remainder Sets

Let us start our discussion with an interesting collection of sets in Diophantine Approximation known as Bounded Remainder Sets (herein referred to as BRS's). Suppose G is a compact, abelian and metrizable group, and let $\beta \in G$. Let $T_\beta : G \to G$ by $T_\beta(x) = x + \beta$ be uniquely ergodic. A measurable set $A \subseteq G$ is a BRS for T_β if

$$\sup_{x \in G} \sup_{N \in \mathbb{N}} \left| \sum_{n=1}^{N} \chi_A(T_\beta^n(x)) - N|A| \right| < \infty.$$

Over the last 100 years, BRS's have been extensively studied in this setting of compact groups, specifically the 1-torus, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (see [16],[17] and [21]) and the s-torus, $\mathbb{T} = (\mathbb{R}/\mathbb{Z})^s$, $s \geq 2$ (see [6], [19], [24], [26] and [32] among others). In February 2014 (see [10]), Haynes and Koivusalo gave a way to construct infinite families of non-trivial parallelotopes which are BRS's, in any dimension. In April 2014, (See [8]), Sigrid Grepstad and Nir Lev used dynamical methods and harmonic analysis to construct parallelotopes, in any dimension and of any possible allowable volume, which are BRS's. This effectively completed the classification of volumes of BRS's for totally irrational toral rotations in any dimension.

In 2016, Haynes and Koivusalo in collaboration with Michael Kelly (see [9]) were able to use the cut and project set method to demonstrate how a simple geometric idea can be used to

construct parallelotopes of all possible volumes which are BRS's for any irrational rotation in any dimension, hence giving a very elegant proof of the construction of Grepstad and Lev. Following this, in 2018, Haynes and Koivusalo along with Joanna Furno extended the above mentioned results and classification of BRS's from the countable family $\mathbb{T}^s$, $s \in \mathbb{N}$, to an uncountable collection of connected, compact subgroups of the adelic torus $\mathbb{A}/\mathbb{Q}$. Their construction involves the use of the cut and project method (see [7]). Following is the main theorem of their paper.

Theorem (Furno, Haynes, Koivusalo, 2018). *Suppose* $\boldsymbol{\alpha} = (\alpha_\infty, \alpha_2, \alpha_3, ...) \in \mathbb{A}$, $\alpha_\infty \notin \mathbb{Q}$ *and* $\overline{\mathbb{Z}\boldsymbol{\alpha}} = \mathbb{A}/\mathbb{Q}$. *Then the collection of all possible volumes of BRS's for* $T_{\boldsymbol{\alpha}}$ *is*

$$\left\{ -\gamma\alpha_\infty + \sum_p \{\gamma\alpha_p\}_p + n \geq 0 \ : \ \gamma \in \mathbb{Q}, n \in \mathbb{Z} \right\},$$

where $\{\cdot\}_p \colon \mathbb{Q}_p \to \mathbb{R}$ *is the p-adic fractional part.*

My work with Haynes and Furno is motivated by the desire to complete this direction of inquiry by giving a construction of BRS's of all possible volumes, for all ergodic rotations on higher dimensional adelic tori, $\mathbb{A}^d/\mathbb{Q}^d$, $d \geq 2$. In our paper ([3]), we do this by giving a simple and explicit construction of polytopal BRS's using the cut and project set method. Our construction involves ideas from dynamical systems and harmonic analysis on the adeles, as well as a geometric argument that reduces the existence argument to the case of an irrational rotation on the torus $\mathbb{R}^d/\mathbb{Q}^d$. The notation required for the proof of the main theorem in our paper are complex and I would like to briefly mention them here before stating it.

For $d \in \mathbb{N}$, the interest is in BRS's for rotations on $X_{\mathcal{P}}^d \cong \mathbb{A}_{\mathcal{P}}^d/\Gamma_{\mathcal{P}}^d$ (taken with the product topology). Since it is convenient to examine all the coordinates at each place at once, we consider $\mathbb{A}_{\mathcal{P}}^d$ as a subset of $\mathbb{R}^d \times \prod_{p \in \mathcal{P}} \mathbb{Q}_p^d$. Thus, we consider elements of the form $\vec{\alpha} = (\vec{\alpha}_\infty, \vec{\alpha}_{p_1}, \vec{\alpha}_{p_2}\ldots)$, where

$$\vec{\alpha}_\infty = (\alpha_{\infty,1}, \alpha_{\infty,2}, \ldots, \alpha_{\infty,d}), \text{ and}$$
$$\vec{\alpha}_p = (\alpha_{p,1}, \alpha_{p,2}, \ldots, \alpha_{p,d}) \text{ for } p \in \mathcal{P}.$$

Rotation by $\vec{\alpha}$ on $X_{\mathcal{P}}^d$ is the map $T_{\vec{\alpha}} : X_{\mathcal{P}}^d \to X_{\mathcal{P}}^d$ defined by $T_{\vec{\alpha}}(\vec{x}) = \vec{x} + \vec{\alpha}$. Our main theorem provides a construction of adelic polytope BRS's for $T_{\vec{\alpha}}$, of all possible volumes, in the generic case when this map is ergodic.

Theorem (Das, Furno, Haynes, 2020). *Suppose* $\mathcal{P}$ *and* $X_{\mathcal{P}}^d$ *are defined as above. Suppose further that* $\vec{\alpha} \in X_{\mathcal{P}}^d$ *and that* $1, \alpha_{\infty,1}, \ldots, \alpha_{\infty,d}$ *are linearly independent over* $\mathbb{Q}$. *Then the set of all volumes of BRS's for* $T_{\vec{\alpha}}$ *is*

$$\left\{ \sum_{j=1}^d \left(\gamma_j \alpha_{\infty,j} - \sum_{p \in \mathcal{P}} \{\gamma_j \alpha_{p,j}\}_p \right) + \eta \geq 0 : \gamma_j \in \Gamma_{\mathcal{P}}, \eta \in \mathbb{Z} \right\},$$

where $\{\cdot\}_p : \mathbb{Q}_p \to \mathbb{R}$ *is the p-adic fractional part. Furthermore, for every volume in this set, there is a BRS for* $T_{\vec{\alpha}}$ *of that volume which is the projection to* $X_{\mathcal{P}}^d$ *of the Cartesian product of a parallelotope in* $\mathbb{R}^d$ *with balls centered at 0 in the p-adic directions (all but finitely many of which have radius 1).*

One important idea in our proof is that the return times of the point $0 \in X_{\mathcal{P}}^{d+1}$, under the action of $T_{\vec{\alpha}}$, to a region $A \subseteq X_{\mathcal{P}}^{d+1}$, are in bijective correspondence with the points of a cut and project set, the projection to $X_{\mathcal{P}}^d$ of a *strip* of a lattice in $X_{\mathcal{P}}^{d+2}$. By choosing A in a way that is compatible with the lattice, we can use the group structure to count the return times to A and ensure that we have a BRS. Of course, what follows is the involvement of technical calculations to verify that all allowable volumes are obtained by this construction.

Gaps problems: The three gap theorem

The next topic of study I concentrated on was the connection between gaps problems and dynamics.

Consider the unit circle $\mathbb{R}/\mathbb{Z}$, which we can also think of as the unit interval $[0, 1]$ with 0 and 1 identified. Now, for fixed $\alpha \in \mathbb{R}$ and $N \in \mathbb{N}$, let $\xi_k = \{k\alpha\}$, where $\{\cdot\}$ denotes the fractional part of $k\alpha$, for all $1 \le k \le N$. Then, the elements of the sequence $\{\xi_k\}_{k=1}^N$, partition the unit circle into N intervals. The lengths of these intervals are exactly the gaps between the elements of the sequence $\{\xi_k\}_{k=1}^N$. We denote the kth gap (the distance between ξ_k and its next neighbour to the right) by $\delta_{k,N}$ and let $g_N(\alpha)$ be the number of distinct elements in the set $\{\delta_{k,N} : 1 \le k \le N\}$. *The Steinhaus conjecture, which is also known as the three gap theorem or the three distance theorem, states that for any real number α and natural number N, we have $g_N(\alpha) \le 3$.*

This was initially proved independently in the late 1950s by Sós [33, 34], Surányi [30], and Świerczkowski [31]. It has since been reproved numerous times and generalized in many ways (see the introductions and bibliographies of [12, 13]). In 2017, Marklof and Strömbergsson ([20]) presented a different approach to prove the classical three gap theorem using the space of two-dimensional Euclidean lattices. The utility of their approach lies in its flexibility for generalization to higher dimensional problems where other techniques do not work well (see [12, 13, 15]).

In my next paper coauthored with Haynes (see [4]), we use an adaptation of the lattice based approach to gaps problems in Diophantine approximation to the adeles, to prove a natural generalization of the three gap theorem for rotations on adelic tori. Before stating our main result, I would like to give a short introduction to the setup. Note that $\mathcal{P}, X_{\mathcal{P}}, \boldsymbol{\alpha}$, and $\mathbb{A}_p$ are defined as in the Introduction. We define gaps as nearest neighbor distances, but first we specify a metric on $X_{\mathcal{P}}$. A natural choice of metric on $\mathbb{A}_{\mathcal{P}}$ is given by

$$
|\boldsymbol{\alpha} - \boldsymbol{\beta}| = \begin{cases} \max\left\{|\alpha_\infty - \beta_\infty|_\infty, \max_{p \in \mathcal{P}} |\alpha_p - \beta_p|_p\right\} & \text{if } |\mathcal{P}| < \infty, \\[3mm] \max\left\{|\alpha_\infty - \beta_\infty|_\infty, \max_{p \in \mathcal{P}} \frac{|\alpha_p - \beta_p|_p}{p}\right\} & \text{if } |\mathcal{P}| = \infty. \end{cases}
$$

This metric induces the usual restricted product topology on $\mathbb{A}_{\mathcal{P}}$, and we use it to define the metric

$$
\|\boldsymbol{\alpha} - \boldsymbol{\beta}\| = \min\{|\boldsymbol{\alpha} - \boldsymbol{\beta} - \boldsymbol{\gamma}| : \gamma \in \Gamma_{\mathcal{P}}\}
$$

on $X_{\mathcal{P}}$, which induces the quotient topology (see [14, 35]).

Given $\boldsymbol{\alpha} \in \mathbb{A}_{\mathcal{P}}$ and $N \in \mathbb{N}$, let

$$
S_N(\boldsymbol{\alpha}) = \{\boldsymbol{\xi}_n = n\boldsymbol{\alpha} + \Gamma_{\mathcal{P}} : 1 \le n \le N\} \subset X_{\mathcal{P}},
$$

and for each $1 \leq n \leq N$ let $\delta_{n,N} = \delta_{n,N}(\boldsymbol{\alpha})$ denote the distance from $\boldsymbol{\xi}_n$ to its nearest neighbor in $S_N(\alpha)$. That is,

$$\delta_{n,N} = \min \left\{ \|\boldsymbol{\xi}_m - \boldsymbol{\xi}_n\| > 0 : 1 \leq m \leq N \right\}.$$

As mentioned above, our interest lies in the number of distinct nearest neighbor distances, which we write as

$$g_N(\boldsymbol{\alpha}) = |\{\delta_{n,N}(\boldsymbol{\alpha}) : 1 \leq n \leq N\}|.$$

The main result of this paper is the following theorem.

Theorem (Das, Haynes, 2021). *Let $\mathcal{P}$ be any non-empty set of prime numbers. For any $\boldsymbol{\alpha} \in X_{\mathcal{P}}$ and $N \in \mathbb{N}$, we have that $g_N(\boldsymbol{\alpha}) \leq 3$. Furthermore, there exist $\boldsymbol{\alpha} \in X_{\mathcal{P}}$ and $N \in \mathbb{N}$ for which $g_N(\boldsymbol{\alpha}) = 3$.*

Initially, I worked by myself and completed the proof of the upper bound for $g_N(\boldsymbol{\alpha})$ in the statement of the theorem, which makes up the first part of the proof of the main theorem ([4, Section 4]) in our paper. We then worked together to complete the proof, for which we needed to show that the bound of 3 in our theorem is best possible, i.e., for any choice of $\mathcal{P}$, there are examples of $\boldsymbol{\alpha} \in X_{\mathcal{P}}$ and $N \in \mathbb{N}$ for which $g_N(\boldsymbol{\alpha}) = 3$. One can definitely use a computer to do this for specific choices of $\mathcal{P}$, but some amount of ingenuity is required to deal with arbitrary $\mathcal{P}$ and construct examples for that case. The last section ([4, Section 5]) of our paper comprises of these examples.

Future work

It is important to note that in the classical three gap theorem, gaps are basically the "nearest neighbor distances to the right (or in a particular fixed direction)" and hence the number of nearest neighbor distances will in general be a less than or equal to the number of gaps. Hence, in their paper ([13]), Haynes and Marklof also consider the problem of choosing a direction (for instance, a cone of angles) when looking for nearest neighbors. This gives a more flexible generalization of the setup of the classical result. As a next step to our proof of the three gap theorem for $\mathbb{A}/\mathbb{Q}$, I am interested in working on a similar "directional" version of the gaps problem in higher dimensions for the adeles. One of the first things to understand towards this will be the notation involved and what the concept of direction looks like in this setting.

In their paper, Sigrid and Grepstad ([8]) also characterize the Riemann measurable BRSs in terms of "equidecomposability" and by constructing invariants with respect to this equidecomposition, they derive explicit conditions for a polytope to be a BRS. On similar lines, I am also interested in extending the work done in [3] to give conditions under which adelic polytopes that are measurable with respect to the Haar measure on $X_{\mathcal{P}}^d$, are BRSs with respected to rotations on higher dimensional adelic tori.

Another topic related to Diophantine Approximation that I have closely studied is **Ostrowski Expansion** of integers. This expansion expresses an integer in terms of the basic parameters associated with its continued fraction expansion. To recall (see [1]), for any irrational number β, consider the simple continued fraction expansion, $\beta = [a_0; a_1, a_2, \dots]$. The integers a_k are known as the partial quotients of β and satisfy $a_k \geq 1$, for all $k \geq 1$. For all $k \geq 0$, the reduced rationals $\frac{p_k}{q_k} = [a_0; a_1, a_2, \dots, a_k]$, obtained by truncating the infinite continued fraction expansion, are called the principal convergents of β. We have the following lemma.

Lemma. *Let $\beta \in \mathbb{R}/\mathbb{Q}$ and let a_k and q_k be defined as above. Then, for every $N \in \mathbb{N}$ there exsits a unique integer $K \geq 0$ such that $q_k \leq n < q_{k+1}$, and a unique sequence $\{c_{k+1}\}_{k=0}^{\infty}$ of integers such that*

$$n = \sum_{k=0}^{\infty} c_{k+1}q_k,$$

$$0 \leq c_1 < a_1 \quad and \quad 0 \leq c_{k+1} \leq a_{k+1} \quad for\ all \quad k \geq 1,$$

$$c_k = 0 \quad whenever \quad c_{k+1} = a_{k+1} \quad with \quad k \geq 1,$$

and

$$c_{k+1} = 0 \quad for\ all \quad k > K.$$

The representation given by this lemma is known as the Ostrowski expansion of the integers. Now, for any $\beta > 1$, the Base-β expansion of any $n \in \mathbb{N}$ and the sum of the digits in the Base-β expansion are respectively given by

$$n = \sum_{i \geq 0} \epsilon_i \beta^{-i} \quad and \quad S_\beta(n) = \sum_{i \geq 0} \epsilon_i,$$

where for all i, $\epsilon_i \in \{0, 1, \ldots, \beta - 1\}$.

A lot of interesting work has been done related to Diophantine approximation, Ostrowski expansions and the sum of digits function. For instance, see [2], [5], [25], [28], and [29], among others. I will now explain a related open problem that was recently brought to my attention and I have been interested in. Using a result proved by R. Salem in 1964 (see [22]) and one of the main theorems of their own paper written in 1973 (see [27]), Senge and Strauss were able to ascertain that the number of integers, the sum of whose digits in each of the bases θ and ϕ lies below a fixed bound, is finite if and only if $\dfrac{\log(\theta)}{\log(\phi)}$ is irrational, i.e.,

$$\#\{n \in \mathbb{N} \colon S_\theta(n) \leq c \text{ and } S_\phi(n) \leq c\} < \infty, \text{ for all } c > 0, \quad \text{if and only if} \quad \frac{\log(\theta)}{\log(\phi)} \in \mathbb{R} \setminus \mathbb{Q}.$$

The problem that we are interested in is to prove an analogous result by replacing the Base-θ and Base-ϕ expansions of n with the Ostrowski expansions of n with respect to any two irrational numbers θ and ϕ. As a first step, it will be helpful to prove a particular case, i.e., to show that the number of integers which have finitely many digits in the Ostrowski expansions with respect to irrational numbers $\theta = \frac{1+\sqrt{5}}{2}$ and $\phi = \frac{3+\sqrt{13}}{2}$ is bounded. In the work that has been done until now, we have been able to prove a couple of important technical lemmas and ascertain that the result does not hold true if $\theta = \frac{1+\sqrt{5}}{2}$ and $\phi = 1 + \sqrt{2}$.

The PhD program has been a platform that has helped me to extensively read quite a few important concepts in the field of Dynamical Systems and Number Theory. In the future, I am interested to work not only on the few problems that I have mentioned here, but also to learn about and work on other related concepts with the goal of proving substantial results that will contribute to the mathematical community.

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