Section 3.4
Multiplication of Matrices

If A is a matrix of size $m \times n$ and B is a matrix of size $n \times p$ then the product $AB$ is defined and is a matrix of size $m \times p$.

So, two matrices can be multiplied if and only if the number of columns in the first matrix is equal to the number of rows in the second matrix.

Example 1: Matrix $A$ is of size $10 \times 7$, matrix $B$ is of size $7 \times 2$ and matrix $C$ is of size $2 \times 10$. Decide if each of the following products are defined. If so, give the size of the product.

<table>
<thead>
<tr>
<th>Product defined?</th>
<th>If so, size?</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. $CB$</td>
<td></td>
</tr>
<tr>
<td>b. $AB$</td>
<td></td>
</tr>
<tr>
<td>c. $BC$</td>
<td></td>
</tr>
<tr>
<td>d. $CA^T$</td>
<td></td>
</tr>
</tbody>
</table>

How to Multiply Two Matrices

The element in the $i$th row and $j$th column of $AB$ is found by multiplying each element in the $i$th row of $A$ by the corresponding element in the $j$th column of $B$ and adding the products.

Example 2: Your stock holdings are given by the row matrix (or vector)

$$A = \begin{pmatrix} GM & IBM & BAC \\ 700 & 400 & 200 \end{pmatrix}$$

At the close of trading on a certain day, the prices (in dollars per share) of these stocks (GM, IBM, BAC, respectively) are

$$B = \begin{pmatrix} 50 \\ 120 \\ 42 \end{pmatrix}$$

What is the total value of your holdings as of that day?
Example 3: The following table displays the average grade in each category for an upper level honors math course with 3 students.

<table>
<thead>
<tr>
<th></th>
<th>Midterm Exam</th>
<th>Final Exam</th>
<th>Homework Avg</th>
<th>Quiz Avg</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amy</td>
<td>85</td>
<td>90</td>
<td>91</td>
<td>99</td>
</tr>
<tr>
<td>Rob</td>
<td>87</td>
<td>85</td>
<td>81</td>
<td>100</td>
</tr>
<tr>
<td>Sally</td>
<td>75</td>
<td>81</td>
<td>80</td>
<td>85</td>
</tr>
</tbody>
</table>

If the midterm is worth 30% and the final exam is worth 35%, the homework average is worth 15%, and the quiz average is worth 20%, what is each student’s course average? Use matrices to calculate the averages. Interpret your answer.

Averages:
Amy          Rob          Sally
Example 4: A company manufactures tables and chairs. The following matrix gives the time requirements for each table and each chair in each of the given departments.

<table>
<thead>
<tr>
<th></th>
<th>Assembly Dept</th>
<th>Finishing Dept</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chair</td>
<td>2 hr</td>
<td>1 hr</td>
</tr>
<tr>
<td>Table</td>
<td>4 hr</td>
<td>2 hr</td>
</tr>
</tbody>
</table>

Furthermore, the company has two manufacturing plants, one in California and the other in Florida. The hourly rates for each department in each state are given in the following matrix.

<table>
<thead>
<tr>
<th></th>
<th>California</th>
<th>Florida</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assembly Dept</td>
<td>$25</td>
<td>$22</td>
</tr>
<tr>
<td>Finishing Dept</td>
<td>$18</td>
<td>$15</td>
</tr>
</tbody>
</table>

Calculate the labor costs for tables and chair in each state.
Example 5: Let \( A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 4 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 & 4 \\ 2 & 0 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} -10 & 9 \\ -6 & 4 \end{pmatrix} \) and \\
\( D = \begin{pmatrix} -1 & 2 & -3 \end{pmatrix} \). Compute, if possible:

a. \( DB \)

b. \( AB \)

c. \( BA^T \)

d. \( A^T D^T \)

Note: In general, matrix multiplication is not commutative – that is, \( AB \neq BA \).
A **square matrix** is a matrix having the same number of rows as columns.

The **identity matrix** is a square matrix that has 1’s along its main diagonal (from the upper left corner to the lower right corner) and 0’s elsewhere. Since an identity matrix the same number of rows as columns, we simply say an identity matrix is of size \( n \).

The **identity matrix** of size \( n \) is given by

\[
I_n = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1
\end{pmatrix}
\]

In general matrix multiplication is not commutative. However, if \( A \) is a square matrix of size \( n \) the identity matrix of size \( n \) has the following property:

\[
AI_n = I_n A = A
\]

Example 6: Let \( G = \begin{pmatrix} 6 & -2 \\ 4 & -8 \end{pmatrix} \).

a. Give the identity matrix of the same size.

b. Show that GI = IG = G.

\[
\begin{pmatrix} 6 & -2 \\ 4 & -8 \end{pmatrix} \begin{pmatrix} 6 & -2 \\ 4 & -8 \end{pmatrix}
\]
**Multiplication Properties of Matrices**

Let $A$, $B$ and $C$ be matrices whose products and sums are defined. Also let $k$ be a scalar.

1. **Associative Property**: $A(BC) = (AB)C$
2. **Associative Property**: $k(AB) = (kA)B$
3. **Distributive Property**: $A(B + C) = AB + AC$

Example 7: Perform the indicated operations.

\[
\begin{bmatrix}
1 & 2 & 4 \\
2 & -2 & -10
\end{bmatrix}
\begin{bmatrix}
-1 & 2 \\
1 & 0
\end{bmatrix}
-2
\begin{bmatrix}
0 & 10 \\
-4 & -12
\end{bmatrix}
\]