The Pinching Theorem

Suppose \( f(x) \), \( g(x) \) and \( h(x) \) are defined in an open interval containing \( x = c \) (except possibly at \( x = c \)) and \( f(x) \leq g(x) \leq h(x) \). If \( \lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L \), then since \( g(x) \) is “pinched” between the functions \( f(x) \) and \( h(x) \), we can conclude that \( \lim_{x \to c} g(x) = L \).

Take \( \lim_{x \to 0} \frac{\sin(x)}{x} \). Direct substitution gives the indeterminate form \( \frac{0}{0} \), but the Pinching allows us to prove that \( \lim_{x \to 0} \frac{\sin(x)}{x} = 1 \).

By applying arithmetic rules to this special limit, we also get \( \lim_{x \to 0} \frac{x}{\sin(x)} = 1 \).

In fact: \( \lim_{x \to 0} \frac{\sin(ax)}{bx} = \frac{a}{b} \)
Example 1: Evaluate.

a. \[ \lim_{x \to 0} \frac{2x}{\sin(5x)} \]

b. \[ \lim_{x \to 0} \frac{15x}{\sin(3x) + 1} \]
c. \[ \lim_{{x \to 0}} \frac{1 - \sec^2(6x)}{(8x)^2} \]
Take \( \lim_{x \to 0} \frac{1 - \cos(x)}{x} \). Direct substitution gives the indeterminate form \( \frac{0}{0} \), but the Pinching allows us to prove that \( \lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0 \).

In fact: \( \lim_{x \to 0} \frac{1 - \cos(ax)}{bx} = 0 \)

Example 2: Evaluate.

a. \( \lim_{x \to 0} \frac{\cos(x) - 1}{\pi x} \)
b. \( \lim_{x \to 0} \frac{1 - \cos^2(2x)}{7x} \)
c. \( \lim_{{x \to 0}} \frac{3x^2}{1 - \cos(5x)} \)
Try these:

\[
\lim_{x \to 0} \frac{\sin(4x)}{\sin(9x)}
\]

\[
\lim_{x \to 0} \frac{\sin^2(3x)}{5x}
\]

\[
\lim_{x \to 0} x^2 \csc(5x^2)
\]

\[
\lim_{x \to \pi} \frac{1 - \cos(3x)}{6}
\]

**Some Useful Trigonometric Identities**

\[\sin^2(\theta) + \cos^2(\theta) = 1\]

\[\tan^2(\theta) + 1 = \sec^2(\theta)\]

\[\cot^2(\theta) + 1 = \csc^2(\theta)\]

\[\sin(2\theta) = 2\sin \theta \cos \theta\]

\[\cos(2\theta) = \cos^2 \theta - \sin^2 \theta\]

\[\cos(2\theta) = 2\cos^2 \theta - 1\]

\[\cos(2\theta) = 1 - 2\sin^2 \theta\]