Section 3.5
Concavity and Points of Inflection

Let $f$ be a function that is differentiable on an open interval $I$.

The graph of $f$ is **concave up** if $f'$ is increasing on $I$.

The graph of $f$ is **concave down** if $f'$ is decreasing on $I$.

Even though both pictures indicate a local extreme value, note that that need not be the case. Here are some graphs where the functions are concave up or down without any local extreme values.

Example 1: The graph of $f'(x)$ (first derivative!) of a polynomial function $f$ is given.

a. When is $f(x)$ concave up? 

b. When is $f(x)$ concave down?
**Theorem:** Let $f$ be a function that is twice differentiable on an open interval $I$.

- If $f''(x) > 0$ for all $x$ in $I$, then the graph of $f$ is concave up on $I$.

- If $f''(x) < 0$ for all $x$ in $I$, then the graph of $f$ is concave down on $I$.

**Determining the Intervals of Concavity for a Function**

1. Find any value of $x$ for which $f''(x) = 0$ or $f''(x)$ is undefined. Identify the intervals determined by these points.

2. Choose a test point $c$ in each interval found in Step 1 and determine the sign of $f''$ in that interval.
   a. Wherever $f''(c) > 0$, then the function $f$ is concave up on that interval.
   b. Wherever $f''(c) < 0$, then the function $f$ is concave down on that interval.

**Example 2:** Determine the concavity of $f(x) = x^3 + 2x$. The domain of $f(x)$ is $(-\infty, \infty)$.

Find when $f''(x) = 0$:

Find when $f''(x)$ is undefined:

**Concave Up:**

**Concave Down:**
A point in the domain of a differentiable function $f$ at which the concavity changes is called a point of inflection.

**Finding Inflection Points**

1. Find any value of $x$ in the domain of the function for which $f''(x) = 0$ or $f''(x)$ is undefined.

2. Determine the sign of $f''(x)$ to the left and to the right of each point $x = a$ found in Step 1. If there is a sign change across the point $x = a$, then $(a, f(a))$ is a point of inflection of $f$.

Example 3: Find any intervals of concavity and points of inflection of $f(x) = 2 + x^{1/3}$. The domain of $f$ is $(-\infty, \infty)$.

Find when $f''(x) = 0$:

Find when $f''(x)$ is undefined:

$\begin{align*}
\text{Concave Up:} & \quad \text{Concave Down:} & \quad \text{POI}
\end{align*}$
Example 4: Given $f(x)$, determine any points of inflection $f(x) = \frac{x}{x^2 - 1}$.

The domain of $f$ is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$ and $f''(x) = \frac{2x(x^2 + 3)}{(x^2 - 1)^3}$.

Find when $f''(x) = 0$:

Find when $f''(x)$ is undefined:

$POI:$

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The graph of $f''$:

When the graph of the second derivative is given, we can gather information about whether $f$ is concave up or down, and any points of inflection for $f$.

Example 5: The graph of $f''$ (second derivative!) of a polynomial function $f$ is given. Determine whether each of the following statements is/are true or false.

a. The function $f(x)$ concave down over one interval.

b. The $x$-values of the points of inflection are: $x = -1, x = 2, x = 5$. 
Sometimes it is difficult to study the sign of the derivative function. For some cases, it may be easier to use the following test:

**The Second-Derivative Test**

Let \( c \) be a critical point for \( f \) where \( f'(c) = 0 \) and \( f''(c) \) exists.

* If \( f''(c) > 0 \), then \( f(c) \) is a local minimum value.
* If \( f''(c) < 0 \), then \( f(c) \) is a local maximum value.
* If \( f''(c) = 0 \), then this test is inconclusive.

Example 6: Given \( f''(x) = 6x - 12 \), \( f'(1) = 0 \) and \( f'(4) = 0 \). Classify these critical numbers as local min/max.
Try this one: Find any critical points and classify them as local min/max on \( \left( 0, \frac{2\pi}{3} \right) \) using the second derivative test.

\[
f(x) = 2\sin x + \cos(2x)
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\[
f'(x) = 2\cos x(1 - 2\sin x)
\]

\[
f''(x) = -2\sin x - 4\cos(2x)
\]