A function is said to be **one-to-one** (1-1) if there are no two distinct numbers in the domain of \( f \) that produce the same value. In other words, two different \( x \) values cannot have the same \( y \) value. If a function has an inverse, then we say it’s **invertible**.

If a function is 1-1, then it has an inverse function, denoted as \( f^{-1} \), which reverses what the first function did. The domain of \( f \) is the range of \( f^{-1} \) and the range of \( f \) is the domain of \( f^{-1} \).

**Example:**  
Celsius to Fahrenheit: \( \frac{9}{5}C + 32 = F \)  
and  
Fahrenheit to Celsius: \( \frac{5}{9}(F - 32) = C \)

**Geometrically**
Property of Inverse Functions

Let \( f \) and \( g \) be two functions such that \( (f \circ g)(x) = x \) for every \( x \) in the domain of \( g \) and \( (g \circ f)(x) = x \) for every \( x \) in the domain of \( f \) then \( f \) and \( g \) are inverses of each other.

Given a function whose graph is known or the given the graph of a function, we can use the Horizontal Line Test to determine if the function is 1-1.

Example 1: Is the following graph of \( f(x) = 2x^{\frac{3}{2}} \) 1-1?

Example 2: Is the function, \( f(x) = (x + x^2)^7 \) 1-1?

Example 3: Is \( f(x) = 3 \sin x \) invertible on \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \)?
A function is **monotonic** if it is always increasing or always decreasing on its domain.

*Recall:*

- If \( f'(x) > 0 \) on its domain, then \( f \) is increasing and; hence, monotonic.
- If \( f'(x) < 0 \) on its domain, then \( f \) is decreasing and; hence, monotonic.

**Theorem:** If \( f \) is monotonic, then \( f \) is an invertible function.

Example 4: Is the following function 1-1? If so, give the equation of the inverse function.

\[
f(x) = \frac{x - 1}{x + 1}
\]

Try this one: Is the following function 1-1? If so, give the equation of the inverse function.

\[
g(x) = x + \frac{4}{x}
\]
Sometimes it’s too long or too difficult to find the equation of the inverse, yet we may want to know if a function has an inverse or not.

Example 5: Is \( f(x) = x^3 + 3x \) invertible?

Example 6: Let \( f(x) = x^3 - kx^2 + 2x \). For what values of \( k \) is \( f(x) \) one-to-one?

Try this one: Let \( f(x) = \frac{1}{3} x^3 - x^2 + kx \). For what values of \( k \) is \( f(x) \) invertible?
Finding the Derivative of the Inverse Function

**Theorem:** If $f(x)$ is continuous and invertible then $f^{-1}(x)$ is continuous.

**Theorem:** If $f(x)$ is differentiable (so must be continuous) and invertible, and $f'(x) \neq 0$, then $f^{-1}(x)$ is differentiable.

If $f(a) = b$ and $f'(a) \neq 0$, then \( (f^{-1})'(b) = \frac{1}{f'(a)} \).

Example 7: For $f(x) = x^3$, we know that $f(2) = 8$. Find $(f^{-1})'(8)$.

Example 8: If $f$ is invertible, and $f(1) = 2$, $f(3) = 1$, $f'(1) = 4$, $f'(3) = 5$, $f'(2) = 6$, find $(f^{-1})'(1)$.
Example 9: Given \( f(x) = x^5 + 1 \), find \( (f^{-1})'(33) \) if possible.

Example 10: If \( f(x) = \sin x + 5 \cos x, \ x \in \left[ 0, \frac{\pi}{2} \right] \), find \( (f^{-1})'(3\sqrt{2}) \).
Example 11: Let \( f(x) = x^3 + 2x^2 + 2x \). The point \((-5, -1)\) is on the graph of \( f^{-1}(x) \). Find \((f^{-1})'(-5)\), then give an equation for the tangent line to the graph of \( f^{-1}(x) \) at the point \((-5, -1)\).