Section 4.1
Inverse Functions

A function is said to be **one-to-one** (1-1) if there are no two distinct numbers in the domain of $f$ that produce the same value. In other words, two different $x$ values cannot have the same $y$ value. If a function has an inverse, then we say it’s **invertible**.

Given a function whose graph is known or given the graph of a function, we can use the Horizontal Line Test to determine if the function is 1-1.

Example 1: Is the function, $f(x) = (x + x^2)^7$ 1-1?

Example 2: Is $f(x) = 3\sin x$ invertible on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$?

If a function is 1-1, then it has an inverse function, denoted as $f^{-1}$, which reverses what the first function did. The domain of $f$ is the range of $f^{-1}$ and the range of $f$ is the domain of $f^{-1}$.

**Geometrically**
**Property of Inverse Functions**

Let $f$ and $g$ be two functions such that $(f \circ g)(x) = x$ for every $x$ in the domain of $g$ and $(g \circ f)(x) = x$ for every $x$ in the domain of $f$ then $f$ and $g$ are inverses of each other.

**Using the Definition to Prove a function is 1-1 or Not**

A function is 1-1 if no two distinct $x$ values give the same image, i.e. If $f(a) = f(b)$ implies $a = b$, the $f$ is 1-1.

Example 3: Prove that $f(x) = x^2 - 4x$ is not 1-1.
Example 4: Prove that \( f(x) = \sqrt[3]{4x - 3} + 2 \) is 1-1. Then find the equation of its inverse.
A function is **monotonic** if it is always increasing or always decreasing on its domain.

*Recall:*

- If \( f'(x) > 0 \) on its domain, then \( f \) is increasing and; hence, monotonic.
- If \( f'(x) < 0 \) on its domain, then \( f \) is decreasing and; hence, monotonic.

**Theorem:** If \( f \) is monotonic, then \( f \) is an invertible function.

Example 5: Is the following function 1-1? If so, give the equation of the inverse function. Then find the domain and range of \( f \) and \( f^{-1} \).

\[
f(x) = \frac{x - 1}{x + 1}
\]
Sometimes it’s too long or too difficult to find the equation of the inverse, yet we may want to know if a function has an inverse or not.

Example 6: Is \( f(x) = x^3 + 3x \) invertible?

Example 7: Let \( f(x) = x^3 - kx^2 + 2x \). For what values of \( k \) is \( f(x) \) one-to-one?

Try this one: Let \( f(x) = \frac{1}{3}x^3 - x^2 + kx \). For what values of \( k \) is \( f(x) \) invertible?
Finding the Derivative of the Inverse Function

Theorem: If $f(x)$ is continuous and invertible then $f^{-1}(x)$ is continuous.

Theorem: If $f(x)$ is differentiable (so must be continuous) and invertible, and $f'(x) \neq 0$, then $f^{-1}(x)$ is differentiable.

If $f(a) = b$ and $f'(a) \neq 0$, then $\left( f^{-1} \right)'(b) = \frac{1}{f'(a)}$.

Example 8: For $f(x) = x^3$, we know that $f(2) = 8$. Find $\left( f^{-1} \right)'(8)$.

Example 9: If $f$ is invertible, and $f(1) = 2$, $f(3) = 1$, $f'(1) = 4$, $f'(3) = 5$, $f'(2) = 6$, find $\left( f^{-1} \right)'(1)$. 
Example 10: Given \( f(x) = x^3 + 1 \), find \( (f^{-1})'(33) \) if possible.

Example 11: If \( f(x) = \sin x + 5 \cos x, \quad x \in \left[0, \frac{\pi}{2}\right] \), find \( (f^{-1})'(3\sqrt{2}) \).
Example 12: Let \( f(x) = x^3 + 2x^2 + 2x \). The point \((-5, -1)\) is on the graph of \( f^{-1}(x) \). Find \( (f^{-1})(-5) \), then give an equation for the tangent line to the graph of \( f^{-1}(x) \) at the point \((-5, -1)\).

Try this one: Is the following function 1-1? If so, give the equation of the inverse function.

\[ g(x) = x + \frac{4}{x} \]