One-to-One Functions and Their Inverses

One-to-One Functions and Their Inverses

Let f be a function with domain A. f is said to be **one-to-one** if no two elements in A have the same image. This means that $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$

if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$

Horizontal Line Test
A function is one-to-one if no horizontal line intersects its graph more than once.

Let f be a one-to-one function with domain A and range B. Then f has an **inverse function**, which is denoted by f^{-1}. The inverse function has domain B and range A and is given by $f^{-1}(y) = x \Leftrightarrow f(x) = y$ for each y in B.

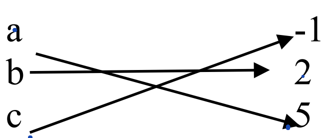
Property of Inverse Functions
$f^{-1}(f(x)) = x$ for each x in A and $f(f^{-1}(x)) = x$ for each x in B.

Let f be a function with domain A. f is said to be one-to-one if no two elements in A have the same image.

Example 1: Determine if the following function is one-to-one.

a.

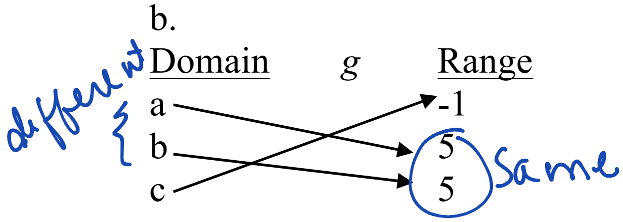
Domain f Range



is one-to-one

b.

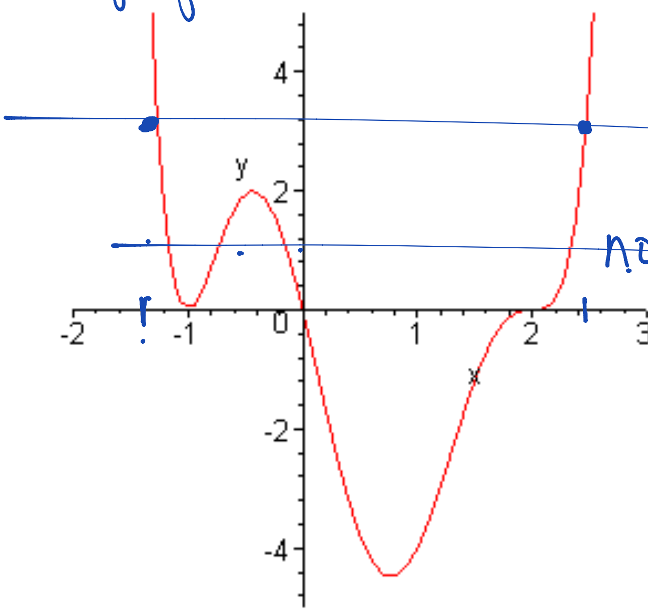
Domain g Range



not one-to-one

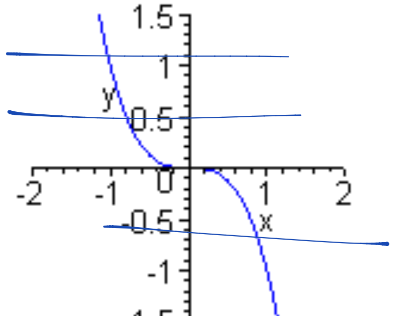
Horizontal line test. determines if function is one-to-one

Vertical line test determines if graph is a function

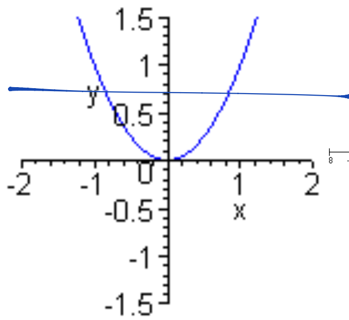


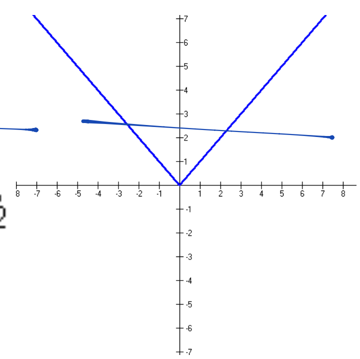
not one-to-one

Which of the following is a one-to- one function



is one to one





not one-to-one

- A one-to-one function has an inverse function. (its inverse is a function)
- The inverse function reverses whatever the first function did.

Example: The formula $f(x) = \frac{9}{5}x + 32$ is used to convert from x degrees Celsius to y degrees Fahrenheit. The formula $g(x) = \frac{5}{9}(x - 32)$ is used to convert from x degrees Fahrenheit to y degrees Celsius

- The inverse of a function f is denoted by f^{-1}, read "f-inverse".
- $f^{-1}(x) \neq \frac{1}{f(x)}$

Example: Assume that the domain of f is all real numbers and that f is one-to-one. If $f(7) = 9$ and $f(8) = -12$

$f^{-1}(9)?$ = 7

$f^{-1}(-12)?$ = 8

Assume that the domain of f is all real numbers and that f is one-to-one. If $f(7) = 17$ and $f(-5) = -11$

$f^{-1}(17)?$ = 7

$f(7) = 17$ same as
$(7, 17)$
$f^{-1}(17) = 7$

If f and g are inverse functions, $f(-2)=3$ and $f(3)=-2$. Find $g(-2)= f^{-1}(-2)=3$

$f(x)=y$

$f^{-1}(y)=x$

If f and g are inverse functions, $f(-1)=-2$ and $f(3)=7$. Find $g(-2)= f^{-1}(-2)=-1$

Domain and Range:

The domain of f is the range of f^{-1} and the range of f is the domain of f^{-1}.

domain & range "switch places" for inverses.

These two statements mean exactly the same thing:

1. f is one-to-one (1-1)
2. f has an inverse function

* Property of Inverse Functions $f(f^{-1}(x)) = f^{-1}(f(x)) = x$

Let f and g be two functions such that $(f \circ g)(x) = x$ for every x in the domain of g and $(g \circ f)(x) = x$ for every x in the domain of f then ***f* and *g* are inverses of each other.**

Example:
Show that the following functions are inverses of each other.

$$f(x) = \frac{3-x}{4} \quad \text{and} \quad g(x) = 3-4x$$

$$f(g(x)) = f(3-4x) = \frac{3-(3-4x)}{4}$$

$$= \frac{3-3+4x}{4}$$

$$= \frac{4x}{4} = x \checkmark$$

$$g(f(x)) = g\left(\frac{3-x}{4}\right) = 3 - \frac{\cancel{4}}{1}\left(\frac{3-x}{\cancel{4}}\right)$$

$$= 3-(3-x)$$

$$= 3-3+x = x \checkmark$$

How to find the inverse of a function: (if it exists!)

1. Replace " $f(x)$ " by "y".
2. Exchange x and y.
3. Solve for y.
4. Replace "y" by " $f^{-1}(x)$ ".
5. Verify! $f(f^{-1}(x)) = f(f^{-1}(x)) = x$

Example:

Find the inverse function of $\cancel{f(x)} = 2x - 7$.

$y = 2x - 7$

$x = 2y - 7$

$+7 \qquad +7$

$\frac{x+7}{2} = \frac{2y}{2}$

$\frac{1}{2}x + \frac{7}{2} = y$

$\boxed{f^{-1}(x) = \frac{1}{2}x + \frac{7}{2}}$

$f(f^{-1}(x)) = f\left(\frac{1}{2}x + \frac{7}{2}\right)$

$= 2\left(\frac{1}{2}x + \frac{7}{2}\right) - 7$

$= x + 7 - 7 = x$ ✓

$f^{-1}(f(x)) = f^{-1}(2x-7)$

$= \frac{1}{2}(2x-7) + \frac{7}{2}$

$= x - \frac{7}{2} + \frac{7}{2} = x$ ✓

Assume $f(x)$ is a one-to-one function. Find the inverse function $f^{-1}(x)$ given that

$\overset{y}{\cancel{f(x)}} = 5x^3 - 2$

$x = 5y^3 - 2$

$+2 \qquad + 2$

$\frac{x+2}{5} = \frac{5y^3}{5}$

$y^3 = \left(\frac{x+2}{5}\right)$

$y = \sqrt[3]{\frac{x+2}{5}}$

$f^{-1}(x) = \sqrt[3]{\frac{x+2}{5}}$

Find the inverse function $f^{-1}(x)$ given that

$$f(x)=\frac{1}{x+2}$$

$$y=\frac{1}{x+2}$$

$$(y+2)x=\frac{1}{(y+2)}(y+2)$$

$$xy+2x=1$$

$$\frac{xy}{x}=\frac{1-2x}{x}$$

$$y=\frac{1-2x}{x}$$

$$\boxed{f^{-1}(x)=\frac{1-2x}{x}}$$

Assume $g(x)$ is a one-to-one function. Find the inverse function $g^{-1}(x)$ given that

$$g(x)=\frac{4x-3}{6-x}$$

$$y=\frac{4x-3}{6-x}$$

$$(6-y)x=\frac{4y-3}{(6-y)}(6-y)$$

$$6x-xy=4y-3$$
$$+xy \qquad +xy$$

$$6x=4y+xy-3$$
$$+3 \qquad +3$$

$$6x+3=4y+xy$$

$$\frac{6x+3}{4+x}=\frac{y(4+x)}{4+x}$$

$$y=\frac{6x+3}{4+x}$$

$$\boxed{g^{-1}(x)=\frac{6x+3}{4+x}}$$

Assume $g(x)$ is a one-to-one function. Find the inverse function $g^{-1}(x)$ given that

$$g(x) = \frac{2}{1-x}$$

$$y = \frac{2}{1-x}$$

$$(1-y)x = \frac{2}{1-y}(1-y)$$

$$x - xy = 2$$
$$-x \qquad\qquad -x$$

$$\frac{-xy}{-x} = \frac{2-x}{-x}$$

$$y = \frac{+1(-2+x)}{+x}$$

$$\boxed{g^{-1}(x) = \frac{x-2}{x}}$$

Verify:

$$g(g^{-1}(x)) = g\left(\frac{x-2}{x}\right)$$

$$= \frac{(2)x}{\left(1 - \frac{x-2}{x}\right)x}$$

$$= \frac{2x}{x-(x-2)} = \frac{2x}{2} = x \checkmark$$

$$g^{-1}(g(x)) = g^{-1}\left(\frac{2}{1-x}\right)$$

$$= \frac{\left(\frac{2}{1-x} - 2\right)(1-x)}{\left(\frac{2}{1-x}\right)(1-x)}$$

$$= \frac{2-2(1-x)}{2}$$

$$= \frac{2-2+2x}{2} = \frac{2x}{2} = x \checkmark$$