Math 1432

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Derivatives and Integrals for Power Series

\[ \sum_{n=0}^{\infty} a_n x^n \]

Expand \( a_n x^n \)

Now, what happens when we take the derivative of this?
Thm – If \( \sum_{n=0}^{\infty} a_n x^n \) converges on (-c, c) then \( \sum_{n=0}^{\infty} \frac{d}{dx}(a_n x^n) \) converges on (-c, c) (you still must check the endpoints for each problem)

Example:

Find the derivative of \( \sum_{n=0}^{\infty} \frac{3nx^n}{n^2 + 1} \)
Integration of Series:

Thm – If \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) converges on \((-c, c)\), then \( g(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \) converges on \((-c, c)\) and \( \int f(x) \, dx = g(x) + C \)

Find a power series for \( \tan^{-1} x \) using integration.
Integrate $\int \sum_{n=0}^{\infty} \frac{3nx^n}{n^2 + 1} \, dx$
(9.8) Definition of nth degree Taylor polynomial centered at $c$:

If $f$ has $n$ derivatives at $c$, then the polynomial

$$P_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x - c)^n$$

is called the nth degree Taylor polynomial for $f$ at $c$. 
Give the 8th degree Taylor polynomial approximation to $y = e^x$ centered at $x = 0$. 

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<thead>
<tr>
<th>$k$</th>
<th>$f^k(x)$</th>
<th>$f^k(0)$</th>
<th>$\frac{f^k(0)}{k!}$</th>
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Find an $n^{\text{th}}$ degree Taylor polynomial approximation for $f(x) = \cos(x)$ centered at $x = 0$.

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Find an $n^{\text{th}}$ degree Taylor polynomial approximation for $f(x) = \sin(x)$ centered at $x = 0$.

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Use the fourth-degree Taylor approximation \( \cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \) for \( x \) near 0 to find \( \lim_{x \to 0} \frac{1 - \cos x}{x^2} \).
1. Give the 7\textsuperscript{th} degree Taylor polynomial approximation for 
   \( f(x) = e^x \) centered at \( x = 0 \).
2. Give the 7th degree Taylor polynomial approximation for \( f(x) = \sin(x) \) centered at \( x = 0 \).
3. Give the 7\textsuperscript{th} degree Taylor polynomial approximation for $f(x) = \cos(x)$ centered at $x = 0$. 

\begin{align*}
a. & x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} \\
b. & 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} \\
c. & 1 - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6} \\
d. & x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \\
\end{align*}
4. Give the coefficient of $x^{10}$ for the 11th degree Taylor polynomial approximation to $\sin(x)$ centered at $x = 0$. 

a. 0 
b. 1 
c. $-\frac{1}{10!}$ 
d. 1
5. Give the coefficient of \((x + 1)^2\) for the 4\(^{th}\) degree Taylor polynomial approximation to \(x^4\) centered at \(x = -1\).

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<tr>
<th>(k)</th>
<th>(f^k(x))</th>
<th>(f^k(-1))</th>
<th>(\frac{f^k(-1)}{k!})</th>
<th>term</th>
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More examples:
1) Find a polynomial of degree $n = 4$ for $f(x) = e^{2x}$ about $x = 0$.

2) Use the Taylor approximation $e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$ for $x$ near 0 to find:

$$\lim_{x \to 0} \frac{e^x - 1}{2x}.$$
3) Use the Taylor approximation $\sin x \approx x - \frac{x^3}{3!}$ for $x$ near 0 to find

$$\lim_{x \to 0} \frac{\sin x}{x}.$$
4) Find the Taylor polynomial of degree \( n = 5 \) for \( f(x) = \ln x \) at \( c = 1 \). Then use \( P_5(x) \) to approximate the value of \( \ln(1.1) \).

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<th>( \frac{f^k(0)}{k!} )</th>
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<td>( f(x) = \ln x )</td>
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<td>( f'(x) = x^{-1} )</td>
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<td>( f'''(x) = 2x^{-3} )</td>
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<td>( f''''(x) = -6x^{-4} )</td>
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<td>( f'''''(x) = 24x^{-5} )</td>
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5) Suppose that $g$ is a function which has continuous derivatives, and that
\[ g(2) = 3, \quad g'(2) = -4, \quad g''(2) = 7, \quad g'''(2) = -5. \]

Write the Taylor polynomial of degree 3 for $g$ centered at $x = 2$. 
6) Find $P_6(x)$ for $f(x) = x^2 \cos(5x)$
7) Find $f^{(15)}(0)$ for $f(x) = e^{x^3}$
Lagrange Form of the Remainder
or
Lagrange Error Bound or Taylor’s Theorem Remainder

When a Taylor polynomial is used to approximate a function, we need a way to see how accurately the polynomial approximates the function.

\[ f(x) = P_n(x) + R_n(x) \quad \text{so} \quad R_n(x) = f(x) - P_n(x) \]

Written in words:

Function = Polynomial + Remainder

\[ f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \ldots \]

so

Remainder = Function – Polynomial
Lagrange Formula for Remainder:

Suppose $f$ has $n+1$ continuous derivatives on an open interval that contains $0$. Let $x$ be in that interval and let $P_n(x)$ be the $n$th Taylor Polynomial for $f$. Then

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

where $c$ is some number between $0$ and $x$.

If we rewrite Taylor’s theorem using the Lagrange formula for the remainder, we have

$$f(x) = f(0) + f'(0) x + \frac{f''(0)}{2!} x^2 + \ldots + \frac{f^{(n)}(0)}{n!} x^n + \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

where $c$ is some number between $0$ and $x$. 
If there is a number $M$ so that $\left| f^{(n+1)}(c) \right| \leq M$

for all $c$ between 0 and $x$ then $\left| f(x) - P_n(x) \right| \leq \frac{M}{(n+1)!} |x|^{n+1}$

or

$\left| R_n(x) \right| \leq \left( \max \left| f^{(n+1)}(c) \right| \right) \frac{|x|^{n+1}}{(n+1)!}$ for $c$ between 0 and $x$.

We probably will not know the value of $c$. 
Give an error estimate for the approximation of \( \sin(x) \) by \( P_9(x) \) for an arbitrary value of \( x \) between 0 and \( \pi/4 \), centered at \( x = 0 \).

\[
\begin{align*}
    f(x) &= \sin x \\
    f'(x) &= \cos x \\
    f''(x) &= -\sin x \\
    f'''(x) &= -\cos x \\
    f^{(4)}(x) &= \sin x
\end{align*}
\]
Give an error estimate for the approximation of $\cos(x)$ by $P_{10}(x)$ for an arbitrary value of $x$ between 0 and $\pi/4$, centered at $x = 0$.

\[ f(x) = \cos x \]
\[ f'(x) = -\sin x \]
\[ f''(x) = -\cos x \]
\[ f'''(x) = \sin x \]
\[ f^{(4)}(x) = \cos x \]
6. Assume that \( f(x) \) is a function such that \( |f^{(10)}(x)| < 15 \) for all \( x \) in the interval (0,1). What is the max possible error for the ninth degree Taylor polynomial centered at 0 for this function when approximating \( f(1) \)?

a. 15
b. 15/9!
c. 15/10!
d. 1

e. none of these