Math 1432

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Office Hours:

Mondays 1-2pm,
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(also available by appointment)

Class webpage:

http://www.math.uh.edu/~bekki/Math1432.html

Notice: If you are in an online lab, you will answer all lab quizzes and submit future homeworks (starting with hw 10) on CASA in Math 1432 section 14151.

Limit Comparison Test:

Know what it does

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Let $\sum a_n$ and $\sum b_n$ be series with positive terms $(a_n > 0, b_n > 0)$ and

$\lim\limits_{n \to \infty} \dfrac{a_n}{b_n} = L$, where $\boldsymbol{L}$ is both **finite and positive**.

Then the two series $\sum a_n$ and $\sum b_n$ either **both** converge or **both**

diverge.

The Limit Comparison Test works well for comparing "messy" algebraic series to a p-series. Choose a p-series with an n^{th} term of the same magnitude as the n^{th} term of the given series.

Examples: Determine whether the following series converge or diverge.

1) $\sum_{n=1}^{\infty} \frac{1}{3n^2 - 4n + 5}$

Know $\sum \frac{1}{n^2}$ conv. (p series w/ $p = 2$)

$$\lim_{n\to\infty} \frac{\frac{1}{3n^2-4n+5}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{3n^2-4n+5} = \frac{1}{3} \Rightarrow \sum \frac{1}{3n^2-4n+5} \text{ conv.}$$

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2) $\sum_{n=1}^{\infty} \frac{n^2 + 10}{4n^3 - n^2 + 7}$

compare to $\sum \frac{1}{n} \to$ diverges (Harmonic Series)

$$\lim_{n\to\infty} \frac{\frac{n^2+10}{4n^3-n^2+7}}{\cancel{1/n}} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^3+10n}{4n^3-n^2+7} = \frac{1}{4}$$

$$\Rightarrow \sum \frac{n^2+10}{4n^3-n^2+7} \text{ diverges}$$

3) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3-2}$ compare w/ $\sum \frac{1}{n^3}$ (conv. p series)

$$\lim_{n\to\infty} \frac{n^3}{n^3-2} = 1$$

converges

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4) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{\sqrt{n^3}+1}$ $n^{1/2}$ compare w/ $\sum \frac{1}{n}$ (div. Harmonic series)

$n^{3/2}$

$$\lim_{n\to\infty} \frac{n^{1/2}}{n^{3/2}+1} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^{3/2}}{n^{3/2}+1} = \underline{1}$$

diverges

5) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n-2}}$ compare w/ $\sum \frac{1}{\sqrt{n}}$ (divergent p series)

$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{3n-2}} = \frac{1}{\sqrt{3}}$ → diverges

Popper 22

1. $\left\{ \frac{1}{\sqrt{n}} \right\}_{n=1}^{\infty}$

a. converges

b. diverges

2. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

a. converges

b. diverges

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^7}$

a. converges

b. diverges

4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+3}$

a. converges

b. diverges

Section 9.5

The Root Test; The Ratio Test

Root Test Let $\sum_{n=b}^{\infty} a_n$ be a series with nonzero terms.

1. $\sum_{n=b}^{\infty} a_n$ converges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$.

2. $\sum_{n=b}^{\infty} a_n$ diverges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$.

3. The Root Test in inconclusive if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$.

   (Use another test.)

The Root Test works well for series involving an *n*th power.

Examples: Determine whether the following series converge or diverge.

1. $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$

$$\lim_{n\to\infty} \sqrt[n]{\frac{e^{2n}}{n^n}} = \lim_{n\to\infty} \frac{e^2}{n} = 0$$

$$0 < 1$$

Converges

2. $\sum_{n=1}^{\infty}\left(\frac{3n+4}{2n}\right)^n$

diverges

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{3n+4}{2n}\right)^n}$$

$$= \lim_{n\to\infty} \frac{3n+4}{2n} = \frac{3}{2} > 1$$

$\lim_{k\to\infty} k^{\frac{1}{k}} =$ 1

1 ← constant

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{3^n}$

Converges

$$\lim_{n\to\infty} \left(\frac{n^3}{3^n}\right)^{1/n} = \lim_{n\to\infty} \frac{n^{3/n}}{3} = \frac{1}{3}$$

$$\frac{1}{3} < 1$$

Ratio Test Let $\sum_{n=b}^{\infty} a_n$ be a series with nonzero terms.

1. $\sum_{n=b}^{\infty} a_n$ converges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$.

2. $\sum_{n=b}^{\infty} a_n$ diverges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$.

3. The Ratio Test is inconclusive if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$

   (Use another test.)

Series involving factorials and exponential functions work especially well in the Ratio Test.

$$(n+1)! = (n+1)\cdot n!$$

$$(n+2)! = (n+2)(n+1)\,n!$$

$$2^{n+1} = 2^n \cdot 2^1$$

Examples: Determine whether the following series converge or diverge.

1. $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

converges

$a_n = \frac{2^n}{n!}$ $\quad a_{n+1} = \frac{2^{n+1}}{(n+1)!}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} a_{n+1} \cdot \frac{1}{a_n}$

$\lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}$

$= \lim_{n\to\infty} \frac{\cancel{2^n} \cdot 2}{(n+1)\cdot \cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{2^n}} = \lim_{n\to\infty} \frac{2}{n+1} = 0$

$0 < 1$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2\, 2^{n+1}}{3^n}$

Converges

$$\lim_{n\to\infty} \frac{(n+1)^2\, 2^{n+2}}{3^{n+1}} \cdot \frac{3^n}{n^2 \cdot 2^{n+1}}$$

$$= \lim_{n\to\infty} \frac{(n^2+2n+1)\, \cancel{2^n} \cdot 2^{\cancel{1}} \cdot \cancel{3^n}}{\cancel{3^n} \cdot 3 \cdot n^2 \cdot \cancel{2^n} \cdot \cancel{2}}$$

$$= \lim_{n\to\infty} \frac{2n^2+4n+2}{3n^2} = \frac{2}{3} < 1$$

3. $\sum_{n=0}^{\infty} \frac{(n+1)!}{3^n}$

diverges

$$\lim_{n\to\infty} \frac{(n+2)!}{3^{n+1}} \cdot \frac{3^n}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{(n+2)\cancel{(n+1)!}\;\cancel{3^n}}{\cancel{3^n}\cdot 3 \cdot \cancel{(n+1)!}}$$

$$= \lim_{n\to\infty} \frac{n+2}{3} \to \infty > 1$$

So here is what we know so far:

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n \neq 0$ ← BDT

$\sum_{n=1}^{\infty} \frac{1}{n}$ Harmonic Series – diverges

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ P-Series – converges if p>1, diverges otherwise

$\sum_{n=0}^{\infty} (r)^n$ Geometric – converges if $|r| < 1$ to $\frac{a_1}{1-r}$ and diverges if $|r| \geq 1$

Basic Comparison Test: $\sum_{n=1}^{\infty} a_n, a_n > 0$

1. If $a_n \geq b_n$ and $\sum_{n=1}^{\infty} b_n, b_n > 0$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges

2. If $a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n, b_n > 0$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

Limit Comparison Test: $\sum_{n=1}^{\infty} a_n, a_n \geq 0$

If you know $\sum_{n=1}^{\infty} b_n, b_n \geq 0$

1. If $\sum_{n=1}^{\infty} b_n$ converges and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ (L is any finite number),then $\sum_{n=1}^{\infty} a_n$ converges

2. If $\sum_{n=1}^{\infty} b_n$ diverges and $\lim_{n\to\infty} \frac{a_n}{b_n} > 0$, then $\sum_{n=1}^{\infty} a_n$ diverges

The Integral Test:
If f is positive, continuous and decreasing for $x \geq 1$ and $a_n = f(n)$, then $\sum_{n=1}^{\infty} a_n$ *and* $\int_1^{\infty} f(x)dx$ either both converge or both diverge.

The Root Test:

Let $\sum a_k$ be a series with nonnegative terms. Suppose $(a_k)^{1/k} \to \rho$, then

1. $\sum a_k$ converges if $\rho < 1$
2. $\sum a_k$ diverges if $\rho > 1$
3. The test is inconclusive if $\rho = 1$

The Ratio Test:

Let $\sum a_k$ be a series with positive terms. Suppose $\frac{a_{k+1}}{a_k} \to \lambda$, then

1. $\sum a_k$ converges if $\lambda < 1$
2. $\sum a_k$ diverges if $\lambda > 1$
3. The test is inconclusive if $\lambda = 1$

5. $\sum_{n=1}^{\infty} 5\underbrace{\cos(n\pi)}$

a. converges

b. diverges

6. $\sum_{n=1}^{\infty} 3n^{-2/3}$ $= 3 \sum \frac{1}{n^{2/3}}$

a. converges

b. diverges

7. $\sum_{n=1}^{\infty} \frac{n+1}{n^3}$ $\sim$ $\sum \frac{1}{n^2}$

$a_n \to 0$?

YES | NO

LOOK Like | div. by BDT

a. converges

b. diverges

8. $\sum_{n=1}^{\infty} \left(\frac{-1}{5} \right)^n$

a. converges

b. diverges