Math 1432

* For 10am
(SW102 - 21456)
next M+W
meet in
100 SEC

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Office Hours:

Mondays 1-2pm,
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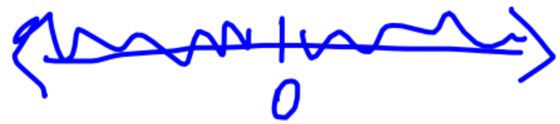
Class webpage:

http://www.math.uh.edu/~bekki/Math1432.html

Find the $\overbrace{\text{radius of convergence}}^{R}$ and interval of convergence for

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!}.$$



for R: take abs value & test for convergence (Root or Ratio)

goal: $|x - c| < R$

$$\left| \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!} \right| \rightarrow \frac{|x|^{2n+1}}{(2n+1)!} \quad \text{Ratio test} \checkmark$$

$$\lim_{n\to\infty} \frac{|x|^{2(n+1)+1}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{|x|^{2n+1}} = \lim_{n\to\infty} \frac{|x|^{2n+3} \cdot (2n+1)!}{(2n+3)!\,|x|^{2n+1}}$$

$$= \lim_{n\to\infty} \frac{\cancel{|x|^{2n}} \cdot |x|^{\cancel{3}\,2} \cdot \cancel{(2n+1)!}}{(2n+3)(2n+2)\cancel{(2n+1)!}\,\cancel{|x|^{2n}} \cdot \cancel{|x|}}$$

$$\lim_{n\to\infty} \frac{|x|^2}{(2n+3)(2n+2)} = 0 \quad \text{always} < 1$$

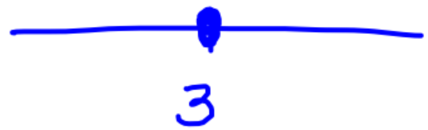
$$\sum \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!} \quad \text{converges for } \underline{\text{all}}\ x$$

$$R = \infty$$

Int. of conv: $(-\infty, \infty)$ $\mathbb{R}$

Find the radius of convergence and interval of convergence for

$$\sum_{n=0}^{\infty} n!(x-3)^n .$$



$R = 0$ int: $\{3\}$

abs value: $n! \cdot |x-3|^n$

ratio: $\displaystyle \lim_{n\to\infty} \frac{(n+1)!\,|x-3|^{n+1}}{n!\,|x-3|^n} = \lim_{n\to\infty} \frac{(n+1)|x-3|}{1}$

$\to \infty$

Derivatives and Integrals for Power Series

Expand $\sum_{n=0}^{\infty} a_n x^n$ $= a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \cdots$

Now, what happens when we take the derivative of this?

$$\frac{d}{dx} \sum_{n=0}^{\infty} a_n x^n = \frac{d}{dx}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right)$$

$$\sum_{n=0}^{\infty} n \cdot a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

Thm – If $\sum_{n=0}^{\infty} a_n x^n$ converges on (-c, c) then $\sum_{n=0}^{\infty} \frac{d}{dx}\left(a_n x^n\right)$ converges on (-c, c) (you still must check the endpoints for each problem)

Example:

Find the derivative of $\sum_{n=0}^{\infty} \frac{3nx^n}{n^2+1}$

$3n \cdot n$

$\downarrow$

$$\frac{d}{dx} \sum_{n=0}^{\infty} \frac{3n x^n}{n^2+1} = \sum_{n=0}^{\infty} \frac{3n^2 x^{n-1}}{n^2+1}$$

Integration of Series:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

Thm – If $f(x)=\sum_{n=0}^{\infty} a_n x^n$ converges on (-c, c), then $g(x)=\sum_{n=0}^{\infty}\frac{a_n}{n+1}x^{n+1}$ converges on (-c, c) and $\int f(x)\,dx = g(x)+C$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Find a power series for $\tan^{-1} x$ using integration.

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n$$

$$= \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

$$\tan^{-1}x = \int \sum_{n=0}^{\infty}(-1)^n x^{2n} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} + C$$

★ $\tan^{-1}(0) = 0$ $\qquad (F(0) = 0)$

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$$

$$f(0) = \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{0}{2n+1}}_{=0} + C = 0$$

$$0 + C = 0 \Rightarrow C = 0$$

so, $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

Integrate $\int \sum_{n=0}^{\infty} \frac{3nx^n}{n^2+1} dx$ $= \sum_{n=0}^{\infty} \frac{3n \quad x^{n+1}}{(n^2+1)(n+1)} + C$

(9.8) Definition of nth degree Taylor polynomial centered at c:

If f has n derivatives at c, then the polynomial

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the nth degree Taylor polynomial for f at c.

Give the 8^{th} degree Taylor polynomial approximation to $y = e^x$ centered at $x = 0$.

$\cdot \ (x-c)^K$

k	$f^k(x)$	$f^k(0)$	$\dfrac{f^k(0)}{k!}$	term
0	e^x	1	$1/0! = 1$	$1\cdot(x-0)^0 = 1$
1	$f'(x) = e^x$	1	1	$1\cdot(x-0)^1 = x$
2	$f''(x) = e^x$	1	$1/2! = 1/2$	$1/2(x-0)^2 = \frac{1}{2}x^2$
3	e^x	1	$1/3!$	$\frac{1}{3!}x^3$
4	e^x	1	$1/4!$	$1/4!\ x^4$
5	e^x	1	$1/5!$	$1/5!\ x^5$
6	e^x	1	$1/6!$	$1/6!\ x^6$
7	e^x	1	$1/7!$	$1/7!\ x^7$
8	e^x	1	$1/8!$	$1/8!\ x^8$

$$P_8(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + x^8/8!$$

Find an n[th] degree Taylor polynomial approximation for $f(x) = \cos(x)$ centered at x = 0.

$(x-0)^k$

k	$f^k(x)$	$f^k(0)$	$\dfrac{f^k(0)}{k!}$	term
0	$\cos x$	1	$1/0! = 1$	1
1	$-\sin x$	0	0	0
2	$-\cos x$	-1	$-1/2!$	$\frac{-1}{2!}(x-0)^2$
3	$\sin x$	0	0	0
4	$\cos x$	1	$1/4!$	$\frac{1}{4!}x^4$
5	$-\sin x$	0	0	0

$$\cos x \rightarrow 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots$$

$$P_{2k}(x) = \sum_{n=0}^{k} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$P_{2n}(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots + \frac{(-1)^n}{(2n)!}x^{2n}$$

degree

Find an nth degree Taylor polynomial approximation for $f(x) = \sin(x)$ centered at x = 0.

$\cdot (x-c)^k$

k	$f^k(x)$	$f^k(0)$	$\frac{f^k(0)}{k!}$	term
0	$\sin x$	0	0	0
1	$\cos x$	1	1	$1 \cdot x$
2	$-\sin x$	0	0	0
3	$-\cos x$	-1	$-1/3!$	$-1/3!\ x^3$
4	$\sin x$	0	0	0
5	$\cos x$	1	$1/5!$	$1/5!\ x^5$

$x - x^3/3! + x^5/5! - x^7/7! + \cdots + (-1)^{n+1}\ x^{2n+1}/(2n+1)!$

Use the fourth-degree Taylor approximation $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ for x near 0 to find $\lim_{x \to 0} \frac{1-\cos x}{x^2}$.

$$1 - \cos x = 1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) = \frac{x^2}{2!} - \frac{x^4}{4!}$$

$$\frac{1-\cos x}{x^2} = \frac{\frac{x^2}{2!} - \frac{x^4}{4!}}{x^2} = \frac{1}{2!} - \frac{x^2}{4!}$$

$$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \lim_{x \to 0} \left(\frac{1}{2!} - \frac{x^2}{4!}\right) = \frac{1}{2}$$

for $\cos(x^2)$ find P_8

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$$

$$\cos(x^2) = 1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \frac{(x^2)^8}{8!}$$

$$P_8(x) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!}$$

Popper 27

1. Give the 7th degree Taylor polynomial approximation for $f(x) = e^x$ centered at x = 0.

a. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7}$

b. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!}$

c. $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$

d. $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

2. Give the 7th degree Taylor polynomial approximation for $f(x) = \sin(x)$ centered at $x = 0$.

a. $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dfrac{x^5}{5} - \dfrac{x^6}{6} + \dfrac{x^7}{7}$

b. $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dfrac{x^5}{5!} + \dfrac{x^6}{6!} + \dfrac{x^7}{7!}$

c. $1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!}$

d. $x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!}$

3. Give the 7th degree Taylor polynomial approximation for $f(x) = \cos(x)$ centered at $x = 0$.

a. $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dfrac{x^5}{5} - \dfrac{x^6}{6} + \dfrac{x^7}{7}$

b. $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dfrac{x^5}{5!} + \dfrac{x^6}{6!} + \dfrac{x^7}{7!}$

c. $1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!}$

d. $x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!}$

4. Give the coefficient of x^{10} for the 11^{th} degree Taylor polynomial approximation to sin (x) centered at $x = 0$.

a. 0

b. $\dfrac{1}{10!}$

c. $\dfrac{-1}{10!}$

d. 1

5. Give the coefficient of $(x + 1)^2$ for the 4^{th} degree Taylor polynomial approximation to x^4 centered at $x = -1$.

a. -6 b. 6 c. -3 d. 3 e. none of these

$\cdot (x-a)^k$

k	$f^k(x)$	$f^k(-1)$	$\dfrac{f^k(-1)}{k!}$	term
0	x^4	1	1	
1	$4x^3$	-4	-4	
2	$12x^2$	12	$\frac{12}{2!} = 6$	$6(x+1)^2$
3	$24x$	-24	$-24/3! = -4$	
4	24	24	$24/4! = 1$	

$$x^4 = 1 - 4(x+1) + 6(x+1)^2 - 4(x+1)^3 + (x+1)^4$$

6. Give the 3rd degree Taylor polynomial for $f(x) = x^3 - 1$ centered at $x = 1$.

a. $3(x-1)+\frac{3}{2}(x-1)^2+\frac{1}{6}(x-1)^3$

b. $3(x-1)+3(x-1)^2+(x-1)^3$

c. $2(x-1)-3(x-1)^2-(x-1)^3$

d. $3(x-1)-3(x-1)^2+(x-1)^3$

e. None of these

7-10=A

K	$f^K(x)$	$f^K(1)$	$\frac{f^K(1)}{K!}$	$(x-1)^K$
0	x^3-1	0	0	0
1	$3x^2$	3	3	$3(x-1)$
2	$6x$	6	3	$3(x-1)^2$
3	6	6	1	