Math 1432

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Office Hours:

Mondays 1-2pm,
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(also available by appointment)

Class webpage:
http://www.math.uh.edu/~bekki/Math1432.html

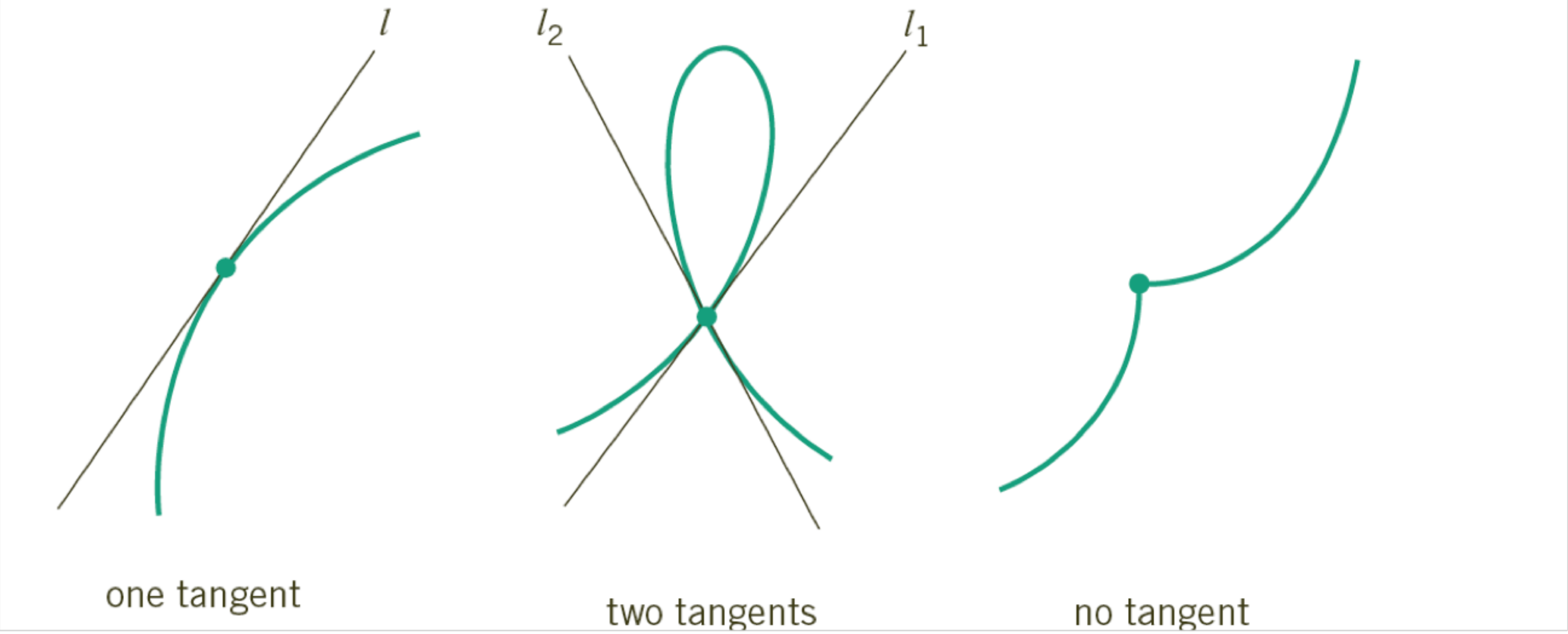
10.4 Derivatives of Curves Given Parametrically

Let C be a curve parametrized by the functions

$$x = x(t), \qquad y = y(t)$$

where x and y are defined on some interval I. Since a curve can intersect itself, at any given point C can have

(i) one tangent, (ii) two or more tangents, or (iii) no tangent at all.



To make sure that at least one tangent line exists at each point of C, we will make the additional *assumption* that

$$\rightarrow \boxed{\left[x'(t)\right]^2+\left[y'(t)\right]^2 \neq 0}$$

This assumption is equivalent to requiring that x'(t) and y'(t) are not simultaneously equal to 0.

If y is a function of t and t is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{\cancel{dt}}\bullet\frac{\cancel{dt}}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \qquad \frac{dx}{dt} \neq 0 \qquad \frac{y'(t)}{x'(t)}$$

Slope of tangent line for parametric curves:

$$m = \frac{dy}{dx} = \frac{y'(t)}{x'(t)}$$

So, the equation of the tangent line to a parametric curve at $\boxed{t = t_0}$ is

$$y - \underline{y(t_0)} = \frac{y'(t_0)}{x'(t_0)}\left(x - \underline{x(t_0)}\right)$$

$$y - y_0 = m(x - x_0)$$

Example: Find $\frac{dy}{dx}$ and evaluate at the given value for

$$x = 2\sqrt{t}, \qquad y = 3t^2 - 2t, \qquad \boxed{t = 1}$$

$$x'(t) = \frac{1}{\sqrt{t}} \qquad y'(t) = 6t - 2$$

$$x'(1) = 1 \qquad y'(1) = 4$$

$$\left.\frac{dy}{dx}\right|_{t=1} = \frac{4}{1} = \boxed{4}$$

Example: Find the **slope and equation of the tangent line** when $\underline{t=1}$;

$F(t)=\left(2t^2, t^3+4t\right)$.

$x(t) = 2t^2$ $\quad x'(t) = 4t$ $\quad x'(1) = 4$ $\quad m = \frac{7}{4}$

$y(t) = t^3+4t$ $\quad y'(t) = 3t^2+4$ $\quad y'(1) = 7$

$x(1) = 2$ $\quad y(1) = 5$

$$y - 5 = \frac{7}{4}(x-2)$$

Exercise: Find an equation for the tangent line when $t=3\pi/4$;

$x(t)=4\sin(t), \quad y(t)=2\tan(t)$.

$x\left(\frac{3\pi}{4}\right) = 4\left(\frac{\sqrt{2}}{2}\right)$ $\quad x'(t) = 4\cos t$ $\quad x'\left(\frac{3\pi}{4}\right) = 4\left(\frac{-\sqrt{2}}{2}\right) = -2\sqrt{2}$

$y\left(\frac{3\pi}{4}\right) = 2(-1)$ $\quad y'(t) = 2\sec^2 t$ $\quad y'\left(\frac{3\pi}{4}\right) = 2\left(-\sqrt{2}\right)^2 = 4$

$m = -4/2\sqrt{2} = -\sqrt{2}$

$$y + 2 = -\sqrt{2}\left(x - 2\sqrt{2}\right)$$

$\longleftrightarrow$ $m = 0$ $\updownarrow$ $m = \emptyset$

Finding Horizontal and Vertical Tangent Lines:

Horiz.: $y'(t) = 0$

Vert: $x'(t) = 0$

(can't have both $= 0$)

Example: $x(t) = t^2, \quad y(t) = t^3 - 3t$

Find the values of t for which the curve has

a) horizontal tangent line(s):

$3t^2 - 3 = 0$

$t^2 - 1 = 0$

$(t-1)(t+1) = 0$ $\rightarrow$ $t = 1, -1$

points:

$(1,-2)\ (1,2)$

$x(1), y(1)$ $\quad$ $x(-1), y(-1)$

$x'(t) = 2t$

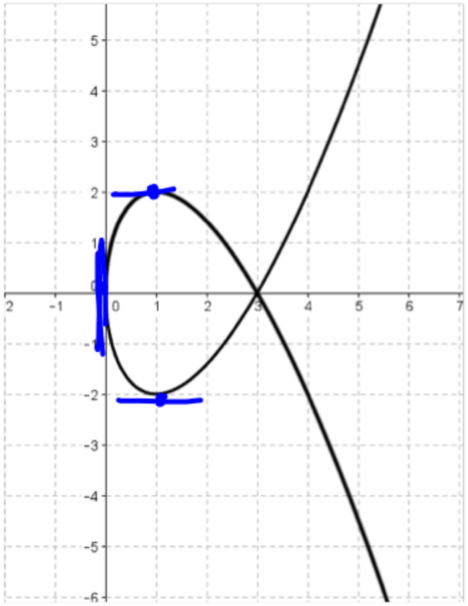
$y'(t) = 3t^2 - 3$

b) vertical tangent line(s):

$2t = 0$

$t = 0$

pt $(0,0)$



Exercise: $x(t) = \sin(2t), \quad y(t) = 4\sin(t)$

Find the values of t for which the curve has

a) horizontal tangent line(s):

$4\cos(t) = 0$

$\cos(t) = 0$

$t = \pi/2, 3\pi/2, \ldots$

$x'(t) = 2\cos(2t)$

$y'(t) = 4\cos t$

b) vertical tangent line(s):

$2\cos(2t) = 0$

$\cos(2t) = 0$

$2t = \pi/2, 3\pi/2, \ldots$

$t = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$

Exercise: Find the **points where the curve has horizontal and vertical tangents:**

$x(t) = 4t^3 - 6t^2, \quad y(t) = t^3 - 12t$

$x'(t) = 12t^2 - 12t = 12t(t-1)$

vert: $t = 0, 1$

$(0,0)$ $(-2,-11)$

$y'(t) = 3t^2 - 12$

$= 3(t^2 - 4) = 3(t-2)(t+2)$

Horiz: $t = 2, -2$

$(8,-16)$ $(-56,16)$

$32 - 24$ $-32 - 24$

$8 - 24$ $-8 + 24$

FINAL EXAM May 7-10 at CASA

Register as soon as possible.

Double check your date and time.

Have someone call you the day of the exam!

Don't forget to bring your ID!

Approx. 20-22 Problems (Some MC, some FR)

Time: 110 minutes

Topics covered: Everything! → 10.4

Practice Final: 5% is added to the final grade

The percentage on the final (without any extra credit) will be used to replace one missed test OR the lowest test (if it is better).

poppers

Go over the class notes, work on past quizzes, EMCFs, practice tests and test reviews.

We will solve some of these problems in class – the rest are exercises for you.

This review sheet is not a complete list of what you need to know. There may be questions on the final that are from topics not included in this sheet. Make sure you take the practice final (several times if necessary). This sheet should not be your only source while studying for the final. GO OVER THE REVIEW SHEETS I POSTED FOR TESTS 2,3, and 4.

1. Evaluate each improper integral and tell why it is improper:

a. $\int_0^{27} x^{-2/3}dx$ $= \lim_{a\to 0+}\int_a^{27} x^{-2/3}dx = \lim_{a\to 0+} 3x^{1/3}\Big|_a^{27}$

$$= \lim_{a\to 0+}\left(3\sqrt[3]{27} - 3\sqrt[3]{a}\right) = 9-0=9$$

b. $\int_0^{4} \frac{1}{\sqrt{4-x}}dx$ $= \lim_{b\to 4-}\int_0^{b} (4-x)^{-1/2}dx = \lim_{b\to 4-}\left[-2(4-x)^{1/2}\right]_0^{b}$

$$= \lim_{b\to 4-}\left(-2\sqrt{4-b} - {}^{-}2\sqrt{4}\right) = 4$$

c. $\int_1^{9} (x-1)^{-2/3}dx$

$$\lim_{a\to 1+}\int_a^{9} (x-1)^{-2/3}dx = \lim_{a\to 1+} 3(x-1)^{1/3}\Big|_a^{9} = \lim_{a\to 1+} 3\sqrt[3]{8} - \underline{3\sqrt[3]{a-1}}$$

$$=6$$

d. $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$

$\lim_{a\to-\infty} \int_{a}^{0} \frac{1}{1+x^2}\,dx + \lim_{b\to\infty} \int_{0}^{b} \frac{1}{1+x^2}\,dx = \lim_{a\to-\infty} \arctan x \Big|_{a}^{0} + \lim_{b\to\infty} \arctan x \Big|_{0}^{b}$

$= \lim_{a\to-\infty} 0 - \tan^{-1} a + \lim_{b\to\infty} \tan^{-1} b$

$= -\left(-\pi/2\right) + \pi/2 = \pi$

2. Integrate (Some may be improper integrals!)

a. $-\frac{1}{3}\int \frac{-3\sin(3x)}{16+\cos^2(3x)}\,dx = -\frac{1}{3}\int \frac{du}{4^2+u^2}$

$u = \cos(3x)$

$du = -3\sin(3x)\,dx$

b. $\int \frac{\overset{3\cdot 2}{6x}}{4+x^4}\,dx = 3\int \frac{du}{2^2+u^2}$

$u = x^2$

$du = 2x\,dx$

$\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$

c. $\displaystyle\int_{0}^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-x^2}}\,dx$

d. $\displaystyle\int \frac{1}{\sqrt{9-4x^2}}\,dx$

e. $\displaystyle\int \frac{5}{36+(x-1)^2}\,dx$

IBP $uv - \int v\,du$

3. Integrate:

a) $\int \underset{A}{2x}\underset{T}{\cos}(10x)\,dx$

$u = 2x \qquad dv = \cos(10x)\,dx$

$du = 2dx \qquad v = \frac{1}{10}\sin(10x)$

$\frac{x}{5}\sin(10x) - \int \frac{1}{5}\sin(10x)\,dx = \frac{x}{5}\sin(10x) + \frac{1}{50}\cos(10x) \underline{\underline{+C}}$

b) $\int 10xe^{4x}\,dx$

c) $\int x\ln(x)\,dx$

$\cos^2 x + \sin^2 x = 1 \quad \tan^2 x + 1 = \sec^2 x \quad \cot^2 x + 1 = \csc^2 x$

4. Integrate:

a) $\int \cos^3 2\theta \, d\theta = \int \cos^2 2\theta \cdot \underline{\cos 2\theta d\theta}$

$\frac{1}{2} \int (1 - \sin^2(2\theta)) \; 2\cos 2\theta d\theta$

$u = \sin(2\theta)$

$du = 2\,\underline{\cos(2\theta) d\theta}$

$\frac{1}{2} \int (1 - u^2)\, du = \frac{1}{2}\left(u - \frac{u^3}{3}\right) + C$

$\frac{1}{2} \sin(2\theta) - \frac{1}{6} \sin^3(2\theta) + C$

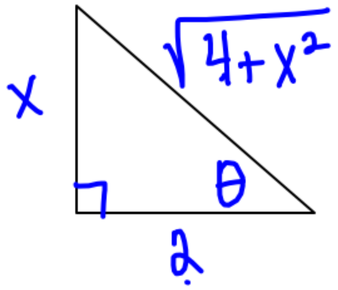
b) $\int \cos^3 \theta \sin^2 \theta \, d\theta$

c) $\int \sec^4 \theta \tan^2 \theta \, d\theta$

5. Integrate:

$$\int \frac{x}{\sqrt{4-x^2}}\,dx$$

$$\int \frac{x^2}{\sqrt{4+x^2}}\,dx$$



$x = 2\tan\theta \qquad (x/2 = \tan\theta)$

$dx = 2\sec^2\theta\, d\theta$

$\sqrt{4+x^2} = 2\sec\theta$

$\downarrow$

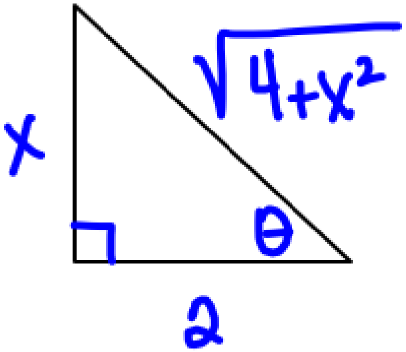
$$\int \frac{(2\tan\theta)^2}{2\sec\theta}\, 2\sec^2\theta\, d\theta = \int 4\tan^2\theta \sec\theta\, d\theta = 4\int \frac{\sin^2\theta}{\cos^2\theta}\cdot\frac{1}{\cos\theta}\,d\theta$$

$$\int 4(\sec^2\theta - 1)\sec\theta\, d\theta$$

$$= 4\int \sec^3\theta - \sec\theta\, d\theta$$

$$4\left(\frac{1}{2}\sec\theta\tan\theta+\frac{1}{2}\ln\left|\sec\theta+\tan\theta\right|-\ln|\sec\theta+\tan\theta\right)+C$$

$$2\sec\theta\tan\theta-2\ln\left|\sec\theta+\tan\theta\right|+C$$



$$2\cdot\left(\frac{\sqrt{4+x^2}}{2}\right)\left(\frac{x}{2}\right)-2\ln\left|\frac{\sqrt{4+x^2}}{2}+\frac{x}{2}\right|+C$$

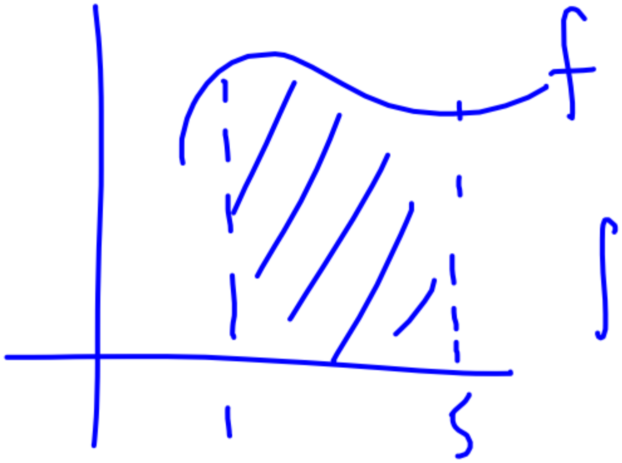
6. Integrate

PFD

$$\int \frac{x^2+5x+2}{(x+1)\left(x^2+1\right)}dx$$

$$\int \frac{2x^2}{(x+1)^2(x-2)}dx$$

7. Let f be a positive function. The area bounded by f(x) and the x-axis from x=1 to x=5 is 21/5. Find the average value of this function.

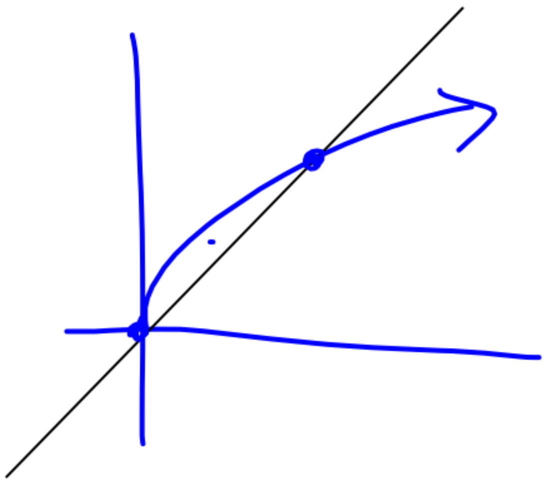


$$AV = \frac{1}{b-a}\underbrace{\int_a^b f(x)\,dx}$$

$$\int_1^5 f(x)\,dx = 21/5$$

$$AV = \frac{1}{5-1}\left(\frac{21}{5}\right) = \frac{21}{20}$$

8. Find the area of the region bounded by $f(x)=\sqrt{x}$ and $g(x)=2x$.



$$\int_0^{1/4}\left(\sqrt{x}-2x\right)dx$$

$\sqrt{x}=2x$

$x=4x^2$

$4x^2-x=0$

$x(4x-1)=0$

$x=0, 1/4$

b) Find the centroid of the region in (a).

9. Given that the centroid of a region is (4,12) and its area is 5, find the volume of the solid formed when this region is rotated about the x-axis.

10. Let R be the region bounded by $f(x) = 3 - x^2$ and $g(x) = 2x$.

Set up the formulas that will give the volume of the solid formed when R is rotated a) about the x-axis.

b) about the y-axis.

c) If R is the base of a solid such that the cross sections perpendicular to x-axis are squares, set up the formula for the volume of that solid.

11. Set up the formula that gives the arc length of the following curve:

a) $f(x)=\frac{2}{3}(x-1)^{3/2}, \quad x\in[1,2]$

b) $r=2+\sin\theta$, in the first quadrant.

12. Solve

$y'=e^{2x}\left(1+y^2\right)$

$$\frac{dy}{dx}=e^{2x}(1+y^2)$$

$$\int\frac{dy}{1+y^2}=\int e^{2x}\,dx$$

13. Given that 10% of a radioactive substance decays in 5 years, give a formula for the amount of substance in terms of t if the initial amount is 100 grams.

14. The population of a bacteria culture increases by 20% in 10 hours. What is the doubling time? What is the population in 24 hours if the initial population is 1000?

15. Determine if the following sequences converge or diverge. If they converge, give the limit.

a. $\left\{(-1)^n\left(\frac{n}{n+1}\right)\right\}$

b. $\left\{\frac{6n^2-2n+1}{4n^2-1}\right\}$

c. $\left\{\frac{(n+2)!}{n!}\right\}$

d. $\left\{\frac{3}{e^n}\right\}$

e. $\left\{\frac{4n+1}{n^2-3n}\right\}$

f. $\left\{\dfrac{e^n}{n^3}\right\}$

*g. $\left\{\dfrac{2n^2+1}{3n^3+4n^2+6}\right\}$

*h. $\left\{\dfrac{1}{n\ln(n)}\right\}$

*i. $\{n\sin(1/n)\}$

*j. $\left\{\left(\dfrac{n-1}{n}\right)^n\right\}$

16. Determine if the following series (A) converge absolutely, (B) converge conditionally or (C) diverge.

a. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n+3}$

b. $\sum_{n=1}^{\infty} \frac{\cos \pi n}{n^2}$

c. $\sum_{n=0}^{\infty} \frac{4n(-1)^n}{3n^2+2n+1}$

d. $\displaystyle\sum_{n=0}^{\infty}\frac{3(-1)^n}{\sqrt{3n^2+2n+1}}$

e. $\displaystyle\sum_{n=0}^{\infty}\frac{3n(-1)^n}{\sqrt{3n^2+2n+1}}$

f. $\displaystyle\sum_{n=0}^{\infty}\left(4(-1)^n\left(\frac{n}{n+3}\right)^n\right)$

g. $$\sum_{n=0}^{\infty}\left(\frac{2(-1)^n \arctan n}{3+n^2+n^3}\right)$$

h. $$\sum_{n=0}^{\infty}\left(\frac{(-1)^n 3^n}{4^n+3n}\right)$$

*j. $$\sum_{n=2}^{\infty}\frac{(-1)^n n!}{(n+1)!}$$

*k. $\sum_{n=2}^{\infty} \frac{(-1)^n}{3n+2}$

*l. $\sum_{n=0}^{\infty} \frac{(-1)^n 10n^2}{3^n}$

*m. $\sum_{n=2}^{\infty} \frac{(-1)^n 3^n}{n!}$

*o. $\sum_{n=2}^{\infty} \frac{\cos(\pi n) n^n}{n!}$

17. Find the sum of the following convergent series:

a. $\sum_{n=0}^{\infty} 2\left(-\frac{4}{9}\right)^{n}$

b. $\sum_{n=0}^{\infty}\left(\frac{1}{3^{n}}-\frac{5}{6^{n}}\right)$

18. Find the radius of convergence and interval of convergence for the following Power series:

a. $\sum_{n=0}^{\infty} \frac{1}{3^n}(x-1)^n$

b. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{4^n}$

19. Give the derivative of each power series below, and for each series, give the antiderivative F of the power series so that F(0)=0.

a. $\sum_{n=0}^{\infty} \frac{(n+1)x^n}{n^2+2}$

$$\int \sum \frac{(n+1)x^n}{n^2+2}\,dx = \sum_{n=0}^{\infty} \frac{\cancel{(n+1)}\,x^{n+1}}{(n^2+2)\cancel{(n+1)}} + C$$

$$0 + \underline{C = 0}$$

b. $\sum_{n=0}^{\infty} \frac{x^n}{2n+1}$

20. Determine the convergence or divergence for each series

Series	Converge or Diverge?	Test used
$\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3}}$		
$\sum_{n=1}^{\infty} \frac{2^n}{n^3}$		
$\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n} \right)$		
$\sum_{n=1}^{\infty} \frac{3^{2n}}{n!}$		
$\sum_{n=1}^{\infty} \cos(\pi n)$		

$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n}$		
$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{3n^3+1}$		
$\sum_{n=0}^{\infty} 3\left(-\frac{1}{2}\right)^n$		
$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$		
$\sum_{n=1}^{\infty} n e^{-n^3}$		
$\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^n$		

$\sum_{n=1}^{\infty} \frac{1}{n^3+1}$		
$\sum_{n=0}^{\infty} \left(\frac{2}{9}\right)^n$		
$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$		
$\sum_{n=2}^{\infty} \frac{10n^2+n-2}{2n^6+7n-1}$		
$\sum_{n=1}^{\infty} \frac{n^2+3n-2}{\sqrt{4n^9+n-1}}$		

21. Suppose $\mathrm{f}(x)=\sum_{n=0}^{\infty}\frac{x^{2n-1}}{(2n)!}$. Give the 13th derivative of f at x = 0.

22. Give the Taylor series expansion for $\mathrm{f}(x)=e^{-x}$ centered at 0.

23. Give Taylor series expansion for $f(x)=\ln(x)$ centered at 1.

K	$f^K(x)$	$f^K(1)$	$f^K(1)/K!$	$(x-1)^K$
0	$\ln x$	0	0	
1	$1/x$	1	1	
2	$-1/x^2$	-1	$-1/2$	
3	$2/x^3$	2	$1/3$	
4	$-6/x^4$	-6	$-1/4$	

$$\sum_{K=1}^{\infty} \frac{(-1)^{K+1}}{K}(x-1)^K$$

24. Give Taylor series expansion for $f(x)=\sin(\underline{3x})$ centered at 0.

25. $f(1)=-1,\ f'(1)=2,\ f''(1)=-1$. Give the 2nd degree Taylor polynomial for f centered at 1.

26. Give a value of n so that the Taylor polynomial of degree n for $\mathrm{f}(x) = \sin(x)$ centered at 0 can be used to approximate $\mathrm{f}(0.5)$ within 10^{-4}.

Know – Graphing polar curves. Converting polar form to rectangular form and vice versa.

Set up the area:

a) Inside one petal of $r = 2\sin 4\theta$.

b) Inside one petal of $r = 4\cos 3\theta$.

c) Inside the inner loop of $r = 1 + 2\sin\theta$

d) Inside the outer loop and to the left of the y-axis, $r = 4 + 8\cos\theta$

e) Inside the curve and to left of the y-axis, $r = 4 + 4\cos\theta$.

27. Find the equation of the tangent and the normal lines to the parametric curves at the given points:

$x(t) = 3\cos(3t) + 2t,\ y(t) = 1 + 5t,$ at $(3,1)$

28. Find the point(s) where the curve has (a) horizontal (b) vertical tangent lines.

$$x(t) = t^2 + 2t, \quad y(t) = 4t^2 + t$$

29. Give an equation relating x and y for the curve given parametrically by

a. $x(t) = -1 + 3\cos t \quad y(t) = 1 + 2\sin t$

c. $x(t) = -1 + 4e^{t} \quad y(t) = 2 + 3e^{-t}$

30. Give a parameterization for the line segment from the point (1, 6) to the point (–3, 1).