Office Hours: Mondays 2-4pm, Thursdays 3-4pm
(also available by appointment)
Office: 639 PGH
Course webpage: www.casa.uh.edu
1. Find the derivative of \( r(t) = \cos(2t)i + \sin(2t)j - \sqrt{5} t k \).

2. Give the norm of \( r'(t) \).
13.3. Curves

The parameterization of a curve represented by the vector-valued function 
\[ r(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \] is smooth on an open interval \( I \) if 
\( x'(t), y'(t), \) and \( z'(t) \) are continuous on \( I \) and \( r'(t) \neq \mathbf{0} \) for any value of \( t \) on the interval \( I \).

Let \( C : r(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \) be a smooth curve. The vector 
\( r'(t) \), if not \( \mathbf{0} \), is tangent to the curve \( C \) at the point \( P(x(t), y(t), z(t)) \). Note that \( r'(t) \) points in the direction of increasing \( t \).
The **unit tangent vector** $T(t)$ at $t$ is defined to be

$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$

And the equation of the tangent line at the point where $t = t_0$ is

$$\mathbf{R}(u) = \mathbf{r}(t_0) + u \left( \mathbf{r}'(t_0) \right)$$

Now, if $T'(t) = \mathbf{0}$ then the unit tangent vector does not change direction. If $T'(t) \neq \mathbf{0}$, then we can find the **principal normal vector**:

$$\mathbf{N}(t) = \frac{T'(t)}{\|T'(t)\|}$$
The plane determined by the unit tangent vector and the principal normal vector is called the **osculating plane**. The normal vector to the osculating plane is given by the **binormal vector**, \( B(t) = T(t) \times N(t) \).

**Figure 13.1.4**
Example 1: Given \( \mathbf{r}(t) = 2 \cos(t) \mathbf{i} + 2 \sin(t) \mathbf{j} + t \mathbf{k} \) with \( t = \frac{\pi}{4} \), find the unit tangent vector, the principal normal vector and the equation of the osculating plane.

Solution:
3. The unit tangent vector is ____ to the principal normal vector.
Other important formulas involving curves:

(1) Two curves \( \mathbf{r}_1(t) \) and \( \mathbf{r}_2(u) \) intersect if there exists numbers \( t \) and \( u \) such that
\[
\mathbf{r}_1(t) = \mathbf{r}_2(u)
\]

(2) Cosine of the angle between two curves: \( \cos \theta = \frac{\mathbf{r}_1'(t) \cdot \mathbf{r}_2'(u)}{\| \mathbf{r}_1'(t) \| \| \mathbf{r}_2'(u) \|} \)
Example:
Find the point at which the curves intersect and find the angle of intersection.

\[ \mathbf{r}_1(t) = e^{3t}\mathbf{i} + 4\sin(t + \pi/2)\mathbf{j} + (t^2 - 1)\mathbf{k} \]

\[ \mathbf{r}_2(u) = u\mathbf{i} + 4\mathbf{j} + (u^2 - 2)\mathbf{k} \]
13.4 and 13.5. Arc Length and Curvature

Recall (from Calculus II), arc length for a parametric curve \((x(t), y(t))\) is given by \(L(C) = \int_{a}^{b} \sqrt{(x'(t))^2 + (y'(t))^2} \, dt\). Now apply this to a curved traced by \((x(t), y(t), z(t))\) and you have \(L(C) = \int_{a}^{b} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt\). And if \(r(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k}\), this can be re-written as \(L(C) = \int_{a}^{b} \|r'(t)\| \, dt\).
We say the path traced by \((x(t), y(t), z(t))\) from 0 to \(t\) is
\[
s = \int_{0}^{t} \sqrt{(x'(u))^2 + (y'(u))^2 + (z'(u))^2} \, du \quad \text{and} \quad s'(t) = \frac{ds}{dt} = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} = \|r'(t)\|.
\]
If \(r(t)\) is twice differentiable, we say \(r'(t) = v(t)\) and \(r''(t) = v'(t) = a(t)\)
In this figure we have a curve through point $P$ with a tangent line $l$ that intersects the $x$-axis at angle $\phi$ then $\kappa = \left| \frac{d\phi}{ds} \right|$ (the magnitude of the rate of change of the angle per unit of arc length) is called the *curvature*.

Figure 13.1.5
If we have the tangent vector $\mathbf{T}$, then $\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\|$

If a curve is given by $y = y(x)$, then

$$\kappa = \frac{|y''|}{[1 + (y')^2]^{3/2}}$$

**proof:**
If a curve is given by \( \mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} \) then we can use

\[
\kappa = \frac{|x'y'' - y'x''|}{[(x')^2 + (y')^2]^{3/2}}
\]

Along a straight line, the curvature is constantly zero and along a circle of radius \( r \), the curvature is constantly \( 1/r \).

If a curve is given by \( \mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k} \), then we can use

\[
\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\| = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}
\]
Example: Calculate the curvature of
\[ r(t) = 2(1 + t)^{3/2}\mathbf{i} + 2(1 - t)^{3/2}\mathbf{j} + \sqrt{2} t \mathbf{k}. \]
Solution:
Popper 02

Interpret $\mathbf{r}(t)$ as the position of a moving object at time $t$ with

$$\mathbf{r}(t) = \cos(2t)\mathbf{i} + \sin(2t)\mathbf{j} - \sqrt{3}\mathbf{k}$$

4. The unit tangent vector is:

5. The principal normal vector is:

6. The curvature is: